STACKABLE GROUPS, TAME FILLING INVARIANTS, AND ALGORITHMIC PROPERTIES OF GROUPS

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Abstract. We introduce a combinatorial property for finitely generated groups called stackable that implies the existence of an inductive procedure for constructing van Kampen diagrams with respect to a canonical finite presentation. We also define algorithmically stackable groups, for which this procedure is an effective algorithm. This property gives a common model for algorithms arising from both rewriting systems and almost convexity for groups.

We also introduce a new pair of asymptotic invariants that are filling inequalities refining the notions of intrinsic and extrinsic diameter inequalities for finitely presented groups. These tame filling inequalities are quasi-isometry invariants, up to Lipschitz equivalence of functions (and, in the case of the intrinsic tame filling inequality, up to choice of a sufficiently large set of defining relators). We show that the radial tameness functions of [12] are equivalent to the extrinsic tame filling inequality condition, and so intrinsic tame filling inequalities can be viewed as the intrinsic analog of radial tameness functions.

We discuss both intrinsic and extrinsic tame filling inequalities for many examples of stackable groups, including groups with a finite complete rewriting system, Thompson's group F, Baumslag-Solitar groups and their iterates, and almost convex groups. We show that the fundamental group of any closed 3-manifold with a uniform geometry is algorithmically stackable using a regular language of normal forms.

Contents

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1. Introduction and definitions

1.1. **Overview.** In geometric group theory, several properties of finitely generated groups have been defined using a language of normal forms together with geometric or combinatorial conditions on the associated Cayley graph, most notably in the concepts of combable groups and automatic groups in which the normal forms satisfy a fellow traveler property. In this paper, in Subsection 1.3, we use a set of normal forms together with another combinatorial property on the Cayley graph of a finitely generated group to define a property which we call stackable. We show that for any stackable group, these combinatorial properties yield a finite presentation for the group (in Lemma 1.5) and an inductive procedure which, upon input of a word in the generators that represents the identity of the group, constructs a van Kampen diagram for that word over this presentation. We also define a notion of algorithmically stackable, which guarantees that this procedure is an effective algorithm, and a notion of regularly stackable, in which the algorithmically stackable structure utilizes a regular language of normal forms. The structure of these van Kampen diagrams for stackable groups differs from the canonical diagrams arising in combable groups, and the stackable property allows a wider spectrum of filling invariant functions (discussed below) and hence applies to wider classes of groups.

Propositions 1.7 and 1.11. *If G is algorithmically stackable over the finite generating set A, then G has solvable word problem, and there is an algorithm which, upon input of a word $w \in A^*$ that represents the identity in G, will construct a van Kampen diagram for w over the stacking presentation.*

This stackable property provides a uniform model for the canonical procedures for building van Kampen diagrams that arise in both the example of groups with a finite complete rewriting system and the example of almost convex groups, as we show in Sections 5.1 and 5.5. The stackable property for a group G also enables computing asymptotic filling invariants for G by an inductive method; we give an illustration of this in Section 4.

Many asymptotic invariants associated to any group G with a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ have been defined using properties of van Kampen diagrams over this presentation. Collectively, these are referred to as filling invariants; an exposition of many of these is given by Riley in [2, Chapter II]. One of the most well-studied filling functions is the isodiametric, or intrinsic diameter, function for G. The adjective "intrinsic" refers to the fact that the distances are measured in the van Kampen diagram It is natural to consider the distance in the Cayley graph, instead, giving an "extrinsic" property, and in [4], Bridson and Riley defined and studied properties of extrinsic diameter functions. It is often easier to compute

upper bounds for these functions, rather than compute exact values; a group satisfies a diameter inequality for a function f if f is an upper bound for the respective diameter function.

In this paper we also introduce refinements of the notions of diameter inequalities, called tame filling inequalities. Essentially, intrinsic and extrinsic diameter inequalities measure the height of the highest peak in van Kampen diagrams, where height refers to (intrinsic or extrinsic) distance from the basepoint of the diagram, while intrinsic and extrinsic tame filling inequalities give a finer measure of the "hilliness" of these diagrams. In order to accomplish this, we consider not only van Kampen diagrams, but also homotopies that "comb" these diagrams; a collection of van Kampen diagrams and homotopies for all of the words representing the identity of the group is a combed filling. In the last part of this Introduction, in Subsection 1.4, we give the details of the definitions of combed fillings and diameter and tame filling inequalities.

In Section 2, we show in Proposition 2.1 that an intrinsic or extrinsic tame filling inequality with respect to a function f implies an intrinsic or extrinsic (respectively) diameter inequality for the function $n \mapsto \lceil f(n) \rceil$. In [4], Bridson and Riley give an example of a finitely presented group G whose (minimal) intrinsic and extrinsic diameter functions are not Lipschitz equivalent. (Two functions $f, g : \mathbb{N} \to \mathbb{N}$ are *Lipschitz equivalent* if there is a constant C such that for all $n \in N$ we have both $f(n) \leq Cg(Cn + C) + C$ and $g(n) \leq Cf(Cn+C)+C$.) While we have not resolved the relationship between tame filling inequalities in general, we give bounds on their interconnections in Theorem 2.2.

Theorem 2.2. *Let G be a finitely presented group with Cayley complex X and combed filling $\mathcal{D}$. Suppose that $j : \mathbb{N} \to \mathbb{N}$ is a nondecreasing function such that for every vertex v of a van Kampen diagram Δ in $\mathcal{D}$, $d_\Delta(*, v) \leq j(d_X(\epsilon, \pi_\Delta(v)))$, and let $\widetilde{j} : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be defined by $\widetilde{j}(n) := j(\lceil n \rceil) + 1$.*

1. *If G satisfies an extrinsic tame filling inequality for the function f with respect to $\mathcal{D}$, then G satisfies an intrinsic tame filling inequality for the function $\widetilde{j} \circ f$.*
2. *If G satisfies an intrinsic tame filling inequality for the function f with respect to $\mathcal{D}$, then G satisfies an extrinsic tame filling inequality for the function $f \circ \widetilde{j}$.*

Our definition of combed filling was also motivated by the concept of tame combing defined by Mihalik and Tschantz [18], and associated radial tame combing functions advanced by Hermiller and Meier [12]. Tame combings are homotopies in the Cayley complex; in Section 3, we recast them into the setting of van Kampen diagrams in the following portion of Proposition 3.6.

Proposition 3.6'. *Let G be a group with a finite symmetrized presentation $\mathcal{P}$. Up to Lipschitz equivalence of nondecreasing functions, the pair $(G, \mathcal{P})$ satisfies an extrinsic tame filling inequality for a function f if and only if $(G, \mathcal{P})$ satisfies a radial tame combing inequality with respect to f.*

Effectively, Proposition 3.6' shows that the intrinsic tame filling inequality is the intrinsic analog of the radial tame combing inequality.

Every group admitting a radial tame combing inequality for a finite-valued function, and hence, by Proposition 3.6', every group admitting a finite-valued extrinsic tame filling inequality function, must also be tame combable as defined in [18]. Although every group admits extrinsic diameter inequalities for finite-valued functions, it is not yet clear whether every finitely presented group admits tame filling inequalities for such functions. Tschantz [22] has conjectured that there is a finitely presented group that does not admit a tame combing, and as a result, that there exists a finitely presented group which admits an extrinsic diameter inequality for a finite-valued function f, but which does not satisfy an extrinsic tame filling inequality for any finite-valued function, and in particular does not satisfy an extrinsic tame filling inequality for any function Lipschitz equivalent to f.

Section 3 also contains Definitions 3.2 and 3.3 of two more asymptotic invariants which we prove to be equivalent to tame filling inequalities in Propositions 3.4 and 3.6. These alternative views are applied in later sections. A fundamental difference between the intrinsic and extrinsic cases arises in this section, in that iterative constructions that glue van Kampen diagrams preserve extrinsic distances, but not necessarily intrinsic distances.

In Section 4, we show that stackable groups admit a stronger inductive procedure, which produces a combed van Kampen diagram for any input word. That is, there is a canonical combed filling associated to a stackable group. The inductive nature of this associated combed filling yields the following.

Theorem 4.2'. *If G is a stackable group, then G admits intrinsic and extrinsic tame filling inequalities for finite-valued functions.*

Theorem 4.3. *If G is an algorithmically stackable group, then G satisfies both intrinsic and extrinsic tame filling inequalities with respect to a recursive function.*

An immediate consequence of Theorem 4.2 and Proposition 3.6 is that every stackable group satisfies the quasi-isometry invariant property of having a tame combing, developed by Mihalik and Tschantz [18]. If Tschantz's conjecture [22] that a non-tame-combable group exists is true, this would then also imply that there is a finitely presented group that does not admit the stackable property with respect to any finite generating set.

In Section 5 we discuss several examples of (classes of) stackable groups, and compute bounds on their tame filling invariants. To begin, in Section 5.1 we consider groups that can be presented by rewriting systems. A *finite complete rewriting system* for a group G consists of a finite set A and a finite set of rules $R \subseteq A^* \times A^*$ such that as a monoid, G is presented by $G = Mon\langle A \mid u = v$ whenever $(u, v) \in R\rangle$, and the rewritings $xuy \to xvy$ for all $x, y \in A^*$ and (u, v) in R satisfy: (1) each $g \in G$ is represented by exactly one word over A that cannot be rewritten, and (2) the (strict) partial ordering $x > y$ if $x \to x_1 \to ... \to x_n \to y$ is well-founded. The *length* of a rewriting rule $u \to v$ in R is the sum of the lengths of the words u and v. The *string growth complexity* function $\gamma : \mathbb{N} \to \mathbb{N}$ associated to this system is defined by $\gamma(n) =$ the maximal length of a word that is a rewriting of a word of length $\leq n$. We use the algorithm of Section 4 to obtain tame filling inequalities in terms of γ in this case.

Theorem 5.1 and **Corollary 5.3.** *Let G be a group with a finite complete rewriting system. Let γ be the string growth complexity function for the associated minimal system and let ζ*

denote the length of the longest rewriting rule for this system. Then G is regularly stackable and satisfies both intrinsic and extrinsic tame filling inequalities for the recursive function $n \mapsto \gamma(\lceil n \rceil + \zeta + 2) + 1$.

This result has potential to reduce the amount of work in searching for finite complete rewriting systems for groups. A choice of partial ordering used in (2) above implies an upper bound on the string growth complexity function. Then given a lower bound on the intrinsic or extrinsic tame filling inequalities or diameter inequalities, this corollary can be used to eliminate partial orderings before attempting to use them (e.g., via the Knuth-Bendix algorithm) to construct a rewriting system. A further paper by the present authors will address this application more fully.

In Section 5.2, we consider Thompson's group F; i.e., the group of orientation-preserving piecewise linear automorphisms of the unit interval for which all linear slopes are powers of 2, and all breakpoints lie in the the 2-adic numbers. Thompson's group F has been the focus of considerable research in recent years, and yet the questions of whether F is automatic or has a finite complete rewriting system are open (see the problem list at [21]). In [6], Cleary, Hermiller, Stein, and Taback show that Thompson's group F is stackable (and their proof can be shown to give an algorithmic stacking), and we note in Section 5.2 that the set of normal forms associated to this stacking is a deterministic context-free language. In [6] the authors also show (after combining their result with Proposition 3.6) that F admits a linear extrinsic tame filling inequality. In Section 5.2 we show that this group also admits a linear intrinsic tame filling inequality, thus refining the result of Guba [11, Corollary 1] that F has a linear intrinsic diameter function.

In the next two subsections of Section 5, we discuss two specific examples of classes of groups admitting finite complete rewriting systems in more detail. We show in Section 5.4 that the Baumslag-Solitar group $BS(1,p)$ with $p \geq 3$ admits an intrinsic tame filling inequality Lipschitz equivalent to the exponential function $n \mapsto p^n$, utilizing the linear extrinsic tame filling inequality for these groups shown in [6]. We note in Section 5.3 that the iterated Baumslag-Solitar groups G_k are examples of regularly stackable groups admitting recursive intrinsic and extrinsic tame filling inequalities. However, applying the lower bound of Gersten [10] on their intrinsic diameter functions, for each natural number $k > 2$ the group G_k does not admit intrinsic or extrinsic tame filling inequalities with respect to a $k-2$-fold tower of exponentials.

Building upon the characterization of Cannon's almost convexity property [5] by a radial tame combing inequality in [12], in Section 5.5 we show the following.

Theorem 5.6. *Let G be a group with finite generating set A, and let $\iota : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ denote the identity function. The following are equivalent:*

1. *The pair (G, A) is almost convex*
2. *There is a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ for G that satisfies an intrinsic tame filling inequality with respect to ι.*
3. *There is a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ for G that satisfies an extrinsic tame filling inequality with respect to ι.*

Moreover, if any of these hold, then G is algorithmically stackable over A.

The properties in Theorem 5.6 are satisfied by all word hyperbolic groups and cocompact discrete groups of isometries of Euclidean space, with respect to every generating set [5]. They are also satisfied by any group G that is shortlex automatic with respect to the generating set A (again this includes all word hyperbolic groups [8, Thms 3.4.5,2.5.1]); for these groups, the set of shortlex normal forms is a regular language. In the proof of Theorem 5.6, the stackable structure constructed for almost convex groups also utilizes the shortlex normal forms. Hence every shortlex automatic group, including every word hyperbolic group, is regularly stackable.

One of the motivations for the definition of automatic groups was to understand the computational properties of fundamental groups of 3-manifolds. However, the fundamental group of a 3-manifold is automatic if and only if its JSJ decomposition does not contain manifolds with a uniform Nil or Sol geometry [8, Theorem 12.4.7]. In contrast, [14] Hermiller and Shapiro have shown that the fundamental group of every closed 3-manifold with a uniform geometry other than hyperbolic must have a finite complete rewriting system, and so combining this result with Theorems 5.1 and 5.6 yields the following.

Corollary 5.8. *If G is the fundamental group of a closed 3-manifold with a uniform geometry, then G is regularly stackable.*

In [16], Kharlampovich, Khoussainov, and Miasnikov introduced the concept of Cayley graph automatic groups, which utilize a fellow-traveling regular set of "normal forms" in which the alphabet for the normal form words is not necessarily a generating set, resulting in a class of groups which includes all automatic groups but also includes many nilpotent and solvable nonautomatic groups. An interesting open question to ask, then, is what relationships, if any, exist between the classes of stackable groups and Cayley graph automatic groups.

In Section 6, we consider tame filling invariants for a class of combable groups.

Corollary 6.3. *If a finitely generated group G admits a quasi-geodesic language of simple word normal forms satisfying a K-fellow traveler property, then G satisfies linear intrinsic and extrinsic tame filling inequalities.*

In particular, all automatic groups over a prefix-closed language of normal forms satisfy the hypotheses of Corollary 6.3. This result refines that of Gersten [9], that combable groups have a linear intrinsic diameter function.

Finally, in Section 7, we prove that tame filling inequalities are quasi-isometry invariants, in the following.

Theorem 7.1. *Suppose that $(G, \mathcal{P})$ and $(H, \mathcal{P}')$ are quasi-isometric groups with finite presentations. If $(G, \mathcal{P})$ satisfies an extrinsic tame filling inequality with respect to f, then $(H, \mathcal{P}')$ satisfies an extrinsic tame filling inequality with respect to a function that is Lipschitz equivalent to f. If $(G, \mathcal{P})$ satisfies an intrinsic tame filling inequality with respect to f, then after adding all relators of length up to a sufficiently large constant to the presentation $\mathcal{P}'$, the pair $(H, \mathcal{P}')$ satisfies an intrinsic tame filling inequality with respect to a function that is Lipschitz equivalent to f.*

1.2. **Notation.**

Throughout this paper, let G be a group with a finite *symmetric* generating set; that is, such that the generating set A is closed under inversion. We will also assume that for each $a \in A$, the element of G represented by a is not the identity ϵ of G. For a word $w \in A^*$, we write w^{-1} for the formal inverse of w in A^*. For words $v, w \in A^*$, we write $v = w$ if v and w are the same word in A^*, and write $v =_G w$ if v and w represent the same element of G.

The group G also has a presentation $\mathcal{P} = \langle A \mid R\rangle$ that is *symmetrized*; that is, such that the generating set A is symmetric, and the set R of defining relations is closed under inversion and cyclic conjugation. Let X be the Cayley 2-complex corresponding to this presentation, whose 1-skeleton $X^1 = \Gamma$ is the Cayley graph of G with respect to A. Let $E(X) = E(\Gamma)$ be the set of 1-cells (i.e., undirected edges) in X^1. By usual convention, for all $g \in G$ and $a \in A$, we consider both the directed edge labeled labeled a from the vertex g to ga and the directed edge labeled a^{-1} from ga to g to have the same underlying undirected CW complex edge between the vertices labeled g and ga. Let $\vec{E}(X) = \vec{E}(\Gamma)$ be the set of these directed edges of X^1.

For an arbitrary word w in A^* that represents the trivial element ϵ of G, there is a *van Kampen diagram* Δ for w with respect to $\mathcal{P}$. That is, Δ is a finite, planar, contractible combinatorial 2-complex with edges directed and labeled by elements of A, satisfying the properties that the boundary of Δ is an edge path labeled by the word w starting at a basepoint vertex $*$ and reading counterclockwise, and every 2-cell in Δ has boundary labeled by an element of R.

Note that although the definition in the previous paragraph is standard, it is a slight abuse of notation, in that the 2-cells of a van Kampen diagram are polygons whose boundaries are labeled by words in A^*, rather than elements of a (free) group. We will also consider the set R of defining relators as a finite subset of $A^* \setminus \{1\}$, where 1 is the empty word. We do not assume that every defining relator is freely reduced, but the freely reduced representative of every defining relator, except 1, must also be in R.

In general, there may be many different van Kampen diagrams for the word w. Also, we do not assume that van Kampen diagrams in this paper are reduced; that is, we allow adjacent 2-cells in Δ to be labeled by the same relator with opposite orientations.

For any van Kampen diagram Δ with basepoint $*$, let $\pi_\Delta : \Delta \to X$ denote the canonical cellular map such that $\pi_\Delta(*) = \epsilon$ and π_Δ maps edges to edges preserving both label and direction.

See for example [3] or [17] for an exposition of the theory of van Kampen diagrams.

1.3. **Stackable groups: Definitions and motivation.**

Our goal is to define a class of groups for which there is an inductive procedure to construct van Kampen diagrams for all words representing the trivial element. Such a collection $\{\Delta_w \mid w \in A^*, w =_G \epsilon\}$ of van Kampen diagrams for a group G over a presentation $\mathcal{P} = \langle A \mid R\rangle$ (where for each w, the diagram Δ_w has boundary label w) is called a *filling* for the pair $(G, \mathcal{P})$.

We begin with a group G together with a finite inverse-closed generating set A of G. Let Γ be the associated Cayley graph. For each $g \in G$ and $a \in A$, let $e_{g,a}$ denote the directed edge in $\vec{E}(\Gamma)$ with initial vertex g, terminal vertex ga, and label a. Whenever $x \in A^*$ and $a \in A$, we also write $e_{x,a} := e_{g,a}$, where g is the element of G represented by x.

For any set $\mathcal{N} = \{y_g \mid g \in G\}$ of normal forms for G (where $y_g \in A^*$ represents the element $g \in G$), we define the set

$$\vec{E}_d = \vec{E}_{d,\mathcal{N}} := \{e_{g,a} \mid \text{either } y_g a = y_{ga} \text{ or } y_g = y_{ga} a^{-1}\} \subseteq \vec{E}(\Gamma)$$

of *degenerate edges*, and let $E_d = E_{d,\mathcal{N}} \subseteq E(\Gamma)$ be the set of undirected edges underlying the directed edges in $\vec{E}_d$. The complementary set

$$\vec{E}_r = \vec{E}_{r,\mathcal{N}} := \vec{E}(\Gamma) \setminus \vec{E}_d$$

will be called the set of *recursive edges*.

Definition 1.1. *A group G is* stackable *with respect to a finite symmetric generating set A if there exist a set $\mathcal{N}$ of normal forms for G over A with normal form 1 for ϵ, a well-founded strict partial ordering $<$ on the set $\vec{E}_r$ of recursive edges, and a constant k, such that whenever $g \in G$, $a \in A$, and $e_{g,a} \in \vec{E}_r$, then there exists a directed path from g to ga in Γ labeled by a word $a_1 \cdots a_n \in A^n$ of length $n \leq k$ satisfying the property that for each $1 \leq i \leq n$, either $e_{ga_1\cdots a_{i-1},a_i} \in \vec{E}_r$ and $e_{ga_1\cdots a_{i-1},a_i} < e_{g,a}$, or else $e_{ga_1\cdots a_{i-1},a_i} \in \vec{E}_d$.*

We call a group G *stackable* if there is a finite generating set for G with respect to which the group is stackable.

Given a group G that is stackable with respect to a generating set A, one can define a function $c : \vec{E}_r \to A^*$ by choosing, for each $e_{g,a} \in \vec{E}_r$, a label $c(e_{g,a}) = a_1 \cdots a_n \in A^*$ of a directed path in Γ satisfying the property above; that is, $c(e_{g,a}) =_G a$, $n \leq k$, and either $e_{ga_1\cdots a_{i-1},a_i} < e_{g,a}$ or $e_{ga_1\cdots a_{i-1},a_i} \in \vec{E}_d$ for each i. The image $c(\vec{E}_r) \subseteq \cup_{n=0}^{k} A^n$ is a finite set. This function will be called a *stacking map*.

On the other hand, given any set $\mathcal{N}$ of normal forms for G over A and any function $c : \vec{E}_r \to A^*$, we can define a a relation $<_c$ on $\vec{E}_r$ as follows. Whenever e', e are both in $\vec{E}_r$ and e' lies in the path in Γ that starts at the initial vertex of e and is labeled by $c(e)$ (where e' is oriented in the same direction as this path), write $e' <_c e$. Let $<_c$ be the transitive closure of this relation. If c is a stacking map obtained from a stackable structure $(\mathcal{N}, <, k)$ for G over A, then the relation $<_c$ is a subset of the well-founded strict partial ordering $<$, and so $<_c$ is also a well-founded strict partial ordering. Moreover, by König's Infinity Lemma, $<_c$ satisfies the property that for each $e \in \vec{E}_r$, there are only finitely many $e'' \in \vec{E}_r$ with $e'' <_c e$.

Definition 1.2. *A* stacking *for a group G with respect to a finite symmetric generating set A is a pair $(\mathcal{N}, c)$ where $\mathcal{N}$ is a set of normal forms for G over A such that the normal form of the identity is the empty word and $c : \vec{E}_r \to A^*$ is a function satisfying*

- **(S1):** *For each $e_{g,a} \in \vec{E}_r$ we have $c(e_{g,a}) =_G a$.*
- **(S2):** *The relation $<_c$ on $\vec{E}_r$ is a strict partial ordering satisfying the property that for each $e \in \vec{E}_r$, there are only finitely many $e'' \in \vec{E}_r$ with $e'' <_c e$.*

(S3): *The image* $c(\vec{E}_r) = \{c(e) \mid e \in \vec{E}_r\} \subset A^*$ *is a finite set.*

(Note that Property (S2) implies that $c(e_{g,a}) \neq_G a$ for all $e_{g,a} \in \vec{E}_r$.)

From the discussion above, the following is immediate.

Lemma 1.3. *A group G is stackable with respect to a finite symmetric generating set A if and only if G admits a stacking with respect to A.*

For a stacking $(\mathcal{N}, c)$, let R_c be the closure of the set $\{c(e_{g,a})a^{-1} \mid e_{g,a} \in \vec{E}_r\}$ under inversion, cyclic conjugation, and free reduction.

Notation 1.4. *For a stacking $(\mathcal{N}, c)$, the function c is the* stacking map*, the set $c(\vec{E}_r)$ is the* stacking image*, the set R_c is the* stacking relation set*, and $<_c$ is the* stacking ordering*.*

In essence, the two equivalent definitions of stackability in Lemma 1.3 are written to display connections to two other properties: The form of Definition 1.1 closely follows Definition 5.5 of almost convexity, and the stacking map in Definition 1.2 gives rise to rewriting operations, which we discuss next.

Starting from a stacking $(\mathcal{N}, c)$ for a group G with generators A, we describe a *stacking reduction procedure* for finding the normal form for the group element associated to any word, by defining a rewriting operation on words over A, as follows. Whenever a word $w \in A^*$ has a decomposition $w = xay$ such that $x, y \in A^*$, $a \in A$, and the directed edge $e_{x,a}$ of Γ lies in $\vec{E}_r$, then we rewrite $w \to xc(e_{x,a})y$. The definition of the stacking ordering $<_c$ says that for every directed edge e' in the Cayley graph Γ that lies along the path labeled $c(e_{x,a})$ from the vertex labeled x, either e' is a degenerate edge in $\vec{E}_d$, or else $e' \in \vec{E}_r$ and $e' <_c e$. Then Property (S2) shows that starting from the word w, there can be at most finitely many rewritings $w \to w_1 \to \cdots \to w_m = z$ until a word z is obtained which cannot be rewritten with this procedure. The final step of the stacking reduction procedure is to freely reduce the word z, resulting in a word w'.

Now $w =_G w'$, and the word w' (when input into this procedure) is not rewritten with the stacking reduction procedure. Write $w' = a_1 \cdots a_n$ with each $a_i \in A$. Then for all $1 \leq i \leq n$, the edge $e_i := e_{a_1 \cdots a_{i-1}, a_i}$ of Γ does not lie in $\vec{E}_r$, and so must be in $\vec{E}_d$. In the case that $i = 1$, this implies that either $y_\epsilon a_1 = y_{a_1}$ or $y_{a_1} a_1^{-1} = y_\epsilon$. Since the normal form of the identity is the empty word, i.e. $y_\epsilon = 1$, we must have $y_{a_1} = a_1$. Assume inductively that $y_{a_1 \cdots a_i} = a_1 \cdots a_i$. The inclusion $e_{i+1} \in \vec{E}_d$ implies that either $a_1 \cdots a_i \cdot a_{i+1} = y_{a_1 \cdots a_{i+1}}$ or $y_{a_1 \cdots a_{i+1}} a_{i+1}^{-1} = a_1 \cdots a_i$. However, the latter equality on words would imply that the final letter $a_{i+1}^{-1} = a_i$, which contradicts the fact that w' is freely reduced. Hence we have that $w' = y_{w'} = y_w$ is in normal form, and moreover every prefix of w' is also in normal form.

That is, we have shown the following.

Lemma 1.5. *Let G be a group with generating set A and let $(\mathcal{N}, c)$ be a stacking for (G, A). If R_c is the associated stacking relation set, then $\langle A \mid R_c \rangle$ is a finite presentation for G. Moreover, the set $\mathcal{N}$ of normal forms of a stacking is closed under taking prefixes.*

We call the presentation $\langle A \mid R_c\rangle$ the *stacking presentation*. In the Cayley 2-complex X corresponding to this presentation, for each edge e labeled a in the set $\vec{E}_r$, the word $c(e)a^{-1}$ is the label of the boundary path for a 2-cell in X, that traverses the reverse of the edge e.

Any prefix-closed set $\mathcal{N}$ of normal forms for G over A yields a maximal tree T in the Cayley graph Γ, namely the set of edges in the paths in Γ starting at ϵ and labeled by the words in $\mathcal{N}$. The associated set $\vec{E}_d$ of degenerate edges is exactly the set $\vec{E}(T)$ of directed edges lying in this tree, and the edges of $\vec{E}_r$ are the edges of Γ that do not lie in the tree T. Each element w of $\mathcal{N}$ must be a *simple word*, meaning that w labels a simple path, that does not repeat any vertices or edges, in the Cayley graph.

We note that our stacking reduction procedure for finding normal forms for words may not be an effective algorithm. To ensure that this process is algorithmic, we would need to be able to recognize, given $x \in A^*$ and $a \in A$, whether or not $e_{x,a} \in \vec{E}_r$, and if so, be able to find $c(e_{x,a})$. If we extend the map c to a function $c' : \vec{E}(\Gamma) \to A^*$ on all directed edges in Γ, by defining $c'(e) := c(e)$ for all $e \in \vec{E}_r$ and $c'(e) := a$ whenever $e \in \vec{E}_d$ and e has label a, then essentially this means that the graph of the function c', as described by the subset

$$S_c := \{(w, a, c'(e_{w,a})) \mid w \in A^*, a \in A\}$$

of $A^* \times A \times A^*$, should be computable. In that case, given any $(w, a) \in A^* \times A$, by enumerating the words z in A^* and checking in turn whether $(w, a, z) \in S_c$, we can find $c'(w, a)$. (Note that the set S_c is computable if and only if the set $\{(w, a, c(e_{w,a})) \mid w \in A^*, a \in A, e_{w,a} \in \vec{E}_r\}$ describing the graph of c is computable. However, using the latter set in the stacking reduction algorithm has the drawback of requiring us to enumerate the finite (and hence enumerable) set $c(\vec{E}_r)$, but we may not have an algorithm to find this set from the stacking.)

Definition 1.6. *A group G is* algorithmically stackable *if G has a finite symmetric generating set A with a stacking $(\mathcal{N}, c)$ such that the set S_c is recursive.*

We have shown the following.

Proposition 1.7. *If G is algorithmically stackable, then G has solvable word problem.*

As with many other algorithmic classes of groups, it is natural to discuss formal language theoretic restrictions on the associated languages, and in particular on the set of normal forms. Computability of the set S_c implies that the set $\mathcal{N}$ is computable as well (since any word $a_1 \cdots a_n \in A^*$ lies in $\mathcal{N}$ if and only if the word is freely reduced and for each $1 \le i \le n$ the tuple $(a_1 \cdots a_{i-1}, a_i, a_i)$ lies in S_c). Many of the examples we consider in Section 5 will satisfy stronger restrictions on the set $\mathcal{N}$.

Definition 1.8. *A group G is* regularly stackable *if G has a finite symmetric generating set A with a stacking $(\mathcal{N}, c)$ such that the set $\mathcal{N}$ is a regular language and the set S_c is recursive.*

Before discussing the details of the inductive procedure for building fillings from stackings, we first reduce the set of diagrams required.

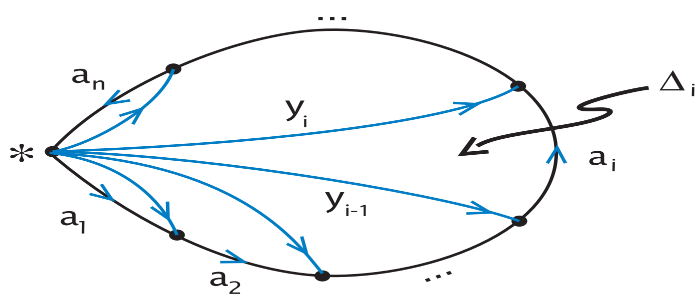

FIGURE 1. Van Kampen diagram built with seashell procedure

For a group G with symmetrized presentation $\mathcal{P} = \langle A \mid R\rangle$ and a set $\mathcal{N} = \{y_g \mid g \in G\} \subseteq A^*$ of normal forms for G, a *normal form diagram* is a van Kampen diagram for a word of the form $y_g a y_{ga}^{-1}$ where $g \in G$ and a in A. We can associate this normal form diagram with the directed edge of the Cayley complex X labeled by a with initial vertex labeled by g. A *normal filling* for the pair $(G, \mathcal{P})$ consists of a set $\mathcal{N}$ normal forms for G that are simple words (i.e. labeling simple paths in the Cayley complex for $\mathcal{P}$), together with a collection $\{\Delta_e \mid e \in E(X)\}$ of normal form diagrams, where for each undirected edge e in X, the normal form diagram Δ_e is associated to one of the two possible directions of e.

Every normal filling induces a filling, using the "seashell" ("cockleshell" in [2, Section 1.3]) method, illustrated in Figure 1, as follows. Given a word $w = a_1 \cdots a_n$ representing the identity of G, with each $a_i \in A$, then for each $1 \le i \le n$, there is a normal form diagram Δ_i in the normal filling that is associated to the edge of X with endpoints labeled by the group elements represented by the words $a_1 \cdots a_{i-1}$ and $a_1 \cdots a_i$. Letting y_i denote the normal form in $\mathcal{N}$ representing $a_1 \cdots a_i$, then the counterclockwise boundary of this diagram is labeled by either $y_{i-1} a_i y_i^{-1}$ or $y_i a_i^{-1} y_{i-1}^{-1}$; by replacing Δ_i by its mirror image if necessary, we may take Δ_i to have counterclockwise boundary word $x_i := y_{i-1} a_i y_i^{-1}$. We next iteratively build a van Kampen diagram Δ_i' for the word $y_\epsilon a_1 \cdots a_i y_i^{-1}$, beginning with $\Delta_1' := \Delta_1$. For $1 < i \le n$, the planar diagrams Δ_{i-1}' and Δ_i have boundary subpaths sharing a common label y_i. The fact that this word is simple, and so labels a simple path in X, implies that any path in a van Kampen diagram labeled by y_i must also be simple, and hence each of these boundary paths is an embedding. These paths are also oriented in the same direction, and so the diagrams Δ_{i-1}' and Δ_i can be glued, starting at their basepoints and folding along these subpaths, to construct the planar diagram Δ_i'. Performing these gluings consecutively for each i results in a van Kampen diagram Δ_n' with boundary label $y_\epsilon w y_w^{-1}$. Note that we have allowed the possibility that some of the boundary edges of Δ_n' may not lie on the boundary of a 2-cell in Δ_n'; some of the words x_i may freely reduce to the empty word, and the corresponding van Kampen diagrams Δ_i may have no 2-cells. Note also that the only simple word representing the identity of G is the empty word; that is, $y_\epsilon = y_w = 1$. Hence Δ_n' is the required van Kampen diagram for w.

Again starting from a stacking $(\mathcal{N}, c)$ for a group G over a finite generating set A, we now give an inductive procedure for constructing a filling for G over the stacking presentation

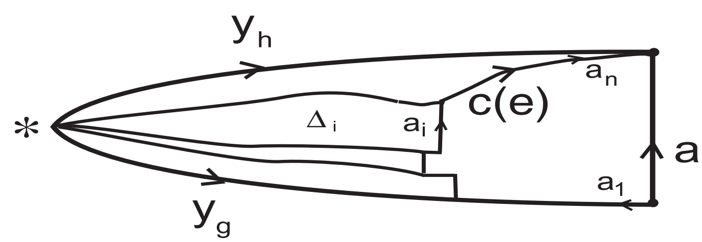

FIGURE 2. Van Kampen diagram Δ_e built from stacking

$\mathcal{P} = \langle A \mid R_c \rangle$ as follows. Let X be the Cayley graph of this presentation. From the argument above, an inductive process for constructing a normal filling from the stacking will suffice. The set of normal forms for the normal filling will be the set $\mathcal{N}$ from the stacking.

We will define a normal form diagram corresponding to each directed edge in $\vec{E}(X) = \vec{E}_r \cup \vec{E}_d$. Let e be an edge in $\vec{E}(X)$, oriented from a vertex g to a vertex h and labeled by $a \in A$, and let $w_e := y_g a y_h^{-1}$.

In the case that e lies in $\vec{E}_d$, the word w_e freely reduces to the empty word. Let Δ_e be the van Kampen diagram for w_e consisting of a line segment of edges, with no 2-cells.

In the case that $e \in \vec{E}_r$, we will use Noetherian induction to construct the normal form diagram. Write $c(e) = a_1 \cdots a_n$ with each $a_i \in A^*$, and for each $1 \leq i \leq n$, let e_i be the edge in X from $ga_1 \cdots a_{i-1}$ to $ga_1 \cdots a_i$ labeled by a_i in the Cayley graph. For each i, either the directed edge e_i is in $\vec{E}_d$, or else $e_i \in \vec{E}_r$ and $e_i <_c e$; in both cases we have, by above or by Noetherian induction, a van Kampen diagram $\Delta_i := \Delta_{e_i}$ with boundary label $y_{ga_1 \cdots a_{i-1}} a_i y_{ga_1 \cdots a_i}^{-1}$. By using the "seashell" method, we successively glue the diagrams Δ_{i-1}, Δ_i along their common boundary words $y_{ga_1 \cdots a_{i-1}}$. Since all of these gluings are along simple paths, this results in a planar van Kampen diagram Δ_e' with boundary word $y_g c(e) y_h^{-1}$. (Note that by our assumption that no generator represents the identity, $c(e)$ must contain at least one letter.) Finally, glue a polygonal 2-cell with boundary label given by the relator $c(e)a^{-1}$ along the boundary subpath $c(e)$ in Δ_e', in order to obtain the diagram Δ_e with boundary word w_e. Since in this step we have glued a disk onto Δ_e' along an arc, the diagram Δ_e is again planar, and is a normal form diagram corresponding to e. (See Figure 2.)

The final step to obtain the normal filling associated to the stacking is to eliminate repetitions. Given any undirected edge e in $E(X)$ choose Δ_e to be a normal form diagram constructed above for one of the orientations of e. Then the collection $\mathcal{N}$ of normal forms, together with the collection $\{\Delta_e \mid e \in E(X)\}$ of normal form diagrams, is a normal filling for the stackable group G.

Definition 1.9. *A* recursive normal filling *is a normal filling that can be constructed from a stacking by the above procedure. A* recursive filling *is a filling induced by a recursive normal filling using seashells.*

Remark 1.10. This recursive normal filling and recursive filling both satisfy a further property which we will exploit in our applications: *For every van Kampen diagram Δ in the filling and every vertex v in Δ, there is an edge path in Δ from the basepoint $*$ to v labeled by the normal form in $\mathcal{N}$ for the element $\pi_\Delta(v)$ in G.*

As with our previous procedure, we have an effective algorithm in the case that the set S_c is computable.

Proposition 1.11. *If G is algorithmically stackable over the finite generating set A, then there is an algorithm which, upon input of a word $w \in A^*$ that represents the identity in G, will construct a van Kampen diagram for w over the stacking presentation.*

Although our stacking reduction procedure above for finding normal forms from a stacking can be used to describe the van Kampen diagrams in this recursive filling more directly, it is this inductive view which will allow us to obtain bounds on filling inequalities for stackable groups in Section 4.

Remark 1.12. For finitely generated groups that are not finitely presented, the concept of a stacking can still be defined, although in this case it makes sense to discuss stackings in terms of a presentation for G, to avoid the (somewhat degenerate) case in which every relator is included in the presentation. A group G with symmetrized presentation $\mathcal{P} = \langle A \mid R \rangle$ is *weakly stackable* if there is a set $\mathcal{N} = \{y_g \mid g \in G\}$ of normal forms over A with $y_\epsilon = 1$ and a function $c : \vec{E}_r \to A^*$ satisfying properties (S1) and (S2) of Definition 1.2 together with the condition that stacking relation set R_c is a subset of R. The pair $(G, \mathcal{P})$ is *algorithmically weakly stackable* if again the set S_c is computable, and *regularly weakly stackable* if the set $\mathcal{N}$ is a regular language and the set S_c is recursive. The stacking reduction procedure and the inductive method for constructing van Kampen diagrams over the presentation $\langle A \mid R_c \rangle$ of G (and hence over $\mathcal{P}$) still hold in this more general setting.

1.4. **Tame filling inequalities: Definitions and motivation.**

Throughout this section, we assume that G is a finitely presented group, with finite symmetrized presentation $\mathcal{P} = \langle A \mid R \rangle$. We begin with a description of the diameter filling inequalities which motivate the tame filling invariants introduced in this paper.

Let X be the Cayley complex for the presentation $\mathcal{P}$, let X^1 be the 1-skeleton of X (i.e., the Cayley graph) and let d_X be the path metric on X^1. Given any word $w \in A^*$, let $l(w)$ denote the length of this word in the free monoid. By slight abuse of notation, $d_X(\epsilon, w)$ then denotes the length of the element of G represented by the word w, where as usual ϵ denotes the identity element of the group G. For any van Kampen diagram Δ with basepoint $*$, let d_Δ denote the path metric on the 1-skeleton Δ^1. Recall that $\pi_\Delta : \Delta \to X$ is the canonical map that maps edges to edges preserving both label and and direction, and satisfies $\pi_\Delta(*) = \epsilon$.

Definition 1.13. *A group G with finite presentation $\mathcal{P}$ satisfies an* intrinsic diameter inequality *for a nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ if for all $w \in A^*$ with $w =_G \epsilon$, there*

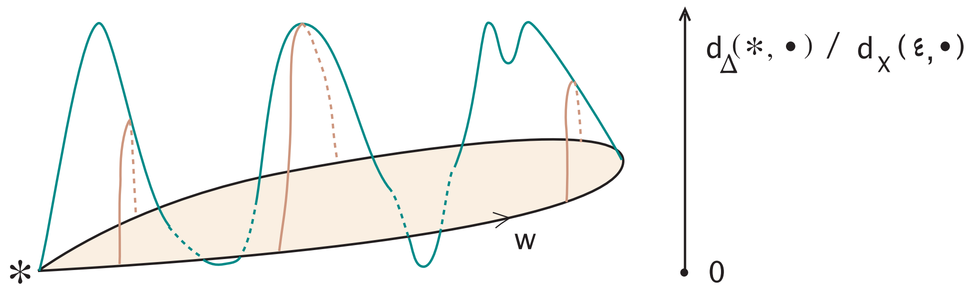

FIGURE 3. Topographic view of van Kampen diagram

exists a van Kampen diagram Δ for w over $\mathcal{P}$ such that for all vertices v in Δ^0 we have $d_\Delta(*, v) \leq f(l(w))$.

The pair $(G, \mathcal{P})$ satisfies an extrinsic diameter inequality *for a nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ if for all $w \in A^*$ with $w =_G \epsilon$, there exists a van Kampen diagram Δ for w over $\mathcal{P}$ such that for all vertices v in Δ^0 we have $d_X(\epsilon, \pi_\Delta(v)) \leq f(l(w))$.*

There are minimal such nondecreasing functions for any pair $(G, \mathcal{P})$, namely the *intrinsic diameter function* or *isodiametric function*, and the *extrinsic diameter function*. See, for example, the exposition in [2, Chapter II] for more details on these two diameter functions.

We build a 3-dimensional view of the 1-skeleton Δ^1 of a van Kampen diagram Δ for a word w by considering the following subsets of $\Delta^1 \times \mathbb{R}$:

$$\begin{aligned} \Delta^i &:= \{(p, d_\Delta(*, p)) \mid p \in \Delta^1\} \\ \Delta^e &:= \{(p, d_X(\epsilon, \pi_\Delta(p))) \mid p \in \Delta^1\} \end{aligned}$$

Viewing the set of points $\Delta^1 \times \{0\}$ at "ground level", and points (p, r) at height r, the sets Δ^i and Δ^e transform the planar van Kampen diagram into a topographical landscape of hills and valleys, as in Figure 3. A pair $(G, \mathcal{P})$ satisfies an intrinsic (resp. extrinsic) diameter inequality for a function $f : \mathbb{N} \to \mathbb{N}$ if for every word w representing the identity of G, there is a van Kampen diagram Δ for w such that the height of the highest peak in Δ^i (resp. Δ^e) is at most $f(l(w)) + \frac{1}{2}$ (the constant $\frac{1}{2}$ takes into account that we have kept the edges in our picture).

These diameter functions are quite coarse, in that they do not measure whether only a few such peaks occur or whether there are many peaks near this maximum height with deep valleys between. Saying this in another way, the diameter functions do not distinguish how wildly or tamely the peaks and valleys occur in van Kampen diagrams. In order to do this, we refine the notion of a diameter function to that of a tame filling function, as follows.

To begin, we define a collection of paths along which we will measure the tameness of the hills. Intuitively, these paths are a continuously chosen "combing" of the boundary of the van Kampen diagram, as illustrated in Figure 4. More formally, we have the following.

Definition 1.14. *A* van Kampen homotopy *of a van Kampen diagram Δ is a continuous function $\Psi : \partial\Delta \times [0, 1] \to \Delta$ satisfying:*

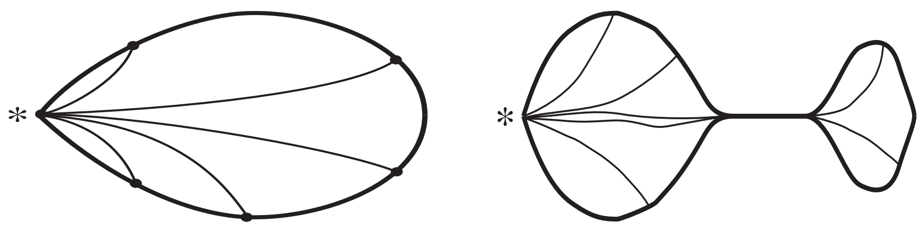

Figure 4. Van Kampen homotopies

(1) *whenever* $p \in \partial\Delta$*, then* $\Psi(p,0) = *$ *and* $\Psi(p,1) = p$*,*
(2) *whenever* $t \in [0,1]$*, then* $\Psi(*,t) = *$*, and*
(3) *whenever* $p \in (\partial\Delta)^0$*, then* $\Psi(p,t) \in \Delta^1$ *for all* $t \in [0,1]$*.*

The diameter inequalities require a filling; that is, a collection of van Kampen diagrams for all words representing ϵ. Analogously, our refinement will require a collection $\{(\Delta_w, \Psi_w) \mid w \in A^*, w =_G \epsilon\}$ such that for each w, Δ_w is a van Kampen diagram with boundary word w, and $\Psi_w : \partial\Delta_w \times [0,1] \to \Delta_w$ is a van Kampen homotopy. We call such a collection a *combed filling* for the pair $(G, \mathcal{P})$.

To streamline notation later, it will be helpful to be able to measure the height (i.e., distance to the basepoint) of each point in the van Kampen diagram, rather than just the height of points in the 1-skeleton. Although we do not necessarily have a metric on a Cayley 2-complex or van Kampen diagram, we can define a coarse notion of distance in any 2-complex Y as follows.

Definition 1.15. *Let* Y *be a combinatorial 2-complex with basepoint vertex* $y \in Y^0$*, and let* p *be any point in* Y*. Define the* coarse distance $\widetilde{d}_Y(y,p)$ *by:*

- *If* p *is a vertex, then* $\widetilde{d}_Y(y,p) := d_Y(y,p)$ *is the path metric distance between the vertices* y *and* p *in the graph* Y^1*.*
- *If* p *is in the interior* $Int(e)$ *of an edge* e *of* Y*, then* $\widetilde{d}_Y(y,p) := \min\{\widetilde{d}_Y(y,v) \mid v \in \partial(e)\} + \frac{1}{2}$*.*
- *If* p *is in the interior of a 2-cell* σ *of* Y*, then* $\widetilde{d}_Y(y,p) := \max\{\widetilde{d}_Y(y,q) \mid q \in Int(e) \text{ for edge } e \text{ of } \partial(\sigma)\} - \frac{1}{4}$*.*

We can now expand the 3-dimensional view above of a van Kampen diagram Δ, by considering the following subsets of $\Delta \times \mathbb{R}$:

$$\begin{aligned}\widetilde{\Delta}^i &:= \{(p, \widetilde{d}_\Delta(*,p)) \mid p \in \Delta\}\\ \widetilde{\Delta}^e &:= \{(p, \widetilde{d}_X(\epsilon, \pi_\Delta(p))) \mid p \in \Delta\}\end{aligned}$$

Our measure of tameness is defined by how high a van Kampen homotopy path can climb in the second coordinate of these sets, and yet still return to a much lower elevation later.

Definition 1.16. *A group* G *with finite presentation* $\mathcal{P}$ *satisfies an* intrinsic tame filling inequality *for a nondecreasing function* $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ *if for all* $w \in A^*$ *with* $w =_G \epsilon$*, there exists a van Kampen diagram* Δ *for* w *over* $\mathcal{P}$ *and a van Kampen homotopy* $\Psi : \partial\Delta \times [0,1] \to \Delta$ *such that*

($\dagger^i$): *for all* $p \in \partial\Delta$ *and* $0 \leq s < t \leq 1$*, we have*

$$\widetilde{d}_\Delta(*, \Psi(p,s)) \leq f(\widetilde{d}_\Delta(*, \Psi(p,t))) \ .$$

In the topographic view, we compose the van Kampen homotopy Ψ with the vertical projection $\nu : \Delta \to \widetilde{\Delta}^i$. Given any point p in the boundary of Δ, the composition $\nu \circ \Psi(p, \cdot) : [0,1] \to \Delta^i$ gives a (discontinuous, in general) path from $(*, 0)$ to $(p, \widetilde{d}_\Delta(*, p))$, i.e. from height 0 to height $\widetilde{d}_\Delta(*, p)$, such that if at any "time" $s \in [0,1]$ the path has reached a height above $f(q)$ for some $q \in \mathbb{N}[\frac{1}{4}]$, then at all later times $t > s$, the path cannot return downward to a height at or below q. Essentially, the tame filling inequality implies that the paths in $\widetilde{\Delta}^i$ rising from the basepoint up to the boundary must go upward steadily, and not keep returning to significantly lower heights.

As with the diameter functions above, we also consider the extrinsic version of this filling function, which has a similar interpretation using the projection to $\widetilde{\Delta}^e$.

Definition 1.17. *A group* G *with finite presentation* $\mathcal{P}$ *satisfies an* extrinsic tame filling inequality *for a nondecreasing function* $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ *if for all* $w \in A^*$ *with* $w =_G \epsilon$*, there exists a van Kampen diagram* Δ *for* w *over* $\mathcal{P}$ *and a van Kampen homotopy* $\Psi : \partial\Delta \times [0,1] \to \Delta$ *such that*

($\dagger^e$): *for all* $p \in \partial\Delta$ *and* $0 \leq s < t \leq 1$*, we have*

$$\widetilde{d}_X(\epsilon, \pi_\Delta(\Psi(p,s))) \leq f(\widetilde{d}_X(\epsilon, \pi_\Delta(\Psi(p,t)))) \ .$$

In contrast to the definition of diameter inequalities, the definition of tame filling inequality does not depend on the length $l(w)$ of the word w. Indeed, the property that a homotopy path $\Psi(p, \cdot)$ cannot return to an elevation below q after it has reached a height above $f(q)$ is uniform for all reduced words over A representing ϵ. As a consequence, it is not clear whether every pair $(G, \mathcal{P})$ has a (intrinsic or extrinsic) tame filling inequality for a finite-valued function.

2. Relationships among filling invariants

The following proposition shows that for finitely presented groups, tameness inequalities imply diameter inequalities.

Proposition 2.1. *If a group* G *with finite presentation* $\mathcal{P}$ *satisfies an intrinsic [resp. extrinsic] tame filling inequality for a nondecreasing function* $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$*, then the pair* $(G, \mathcal{P})$ *also satisfies an intrinsic [resp. extrinsic] diameter inequality for the function* $\hat{f} : \mathbb{N} \to \mathbb{N}$ *defined by* $\hat{f}(n) = \lceil f(n) \rceil$*.*

Proof. We prove this for intrinsic tameness; the extrinsic proof is similar. Let w be any word over the generating set A of the presentation $\mathcal{P}$ representing the trivial element ϵ of G, and let Δ, Ψ be a van Kampen diagram and homotopy for w satisfying the condition ($\dagger^i$). Since the function Ψ is continuous, each vertex $v \in \Delta^0$ satisfies $v = \Psi(p, s)$ for some $p \in \partial\Delta$ and $s \in [0,1]$. There is an edge path along $\partial\Delta$ from $*$ to p labeled by at most half of the word w, and so $\widetilde{d}_\Delta(*, p) \leq \frac{l(w)}{2}$. Using the facts that $p = \Psi(p, 1)$ and $s \leq 1$, condition ($\dagger^i$) implies that $\widetilde{d}_\Delta(*, v) \leq f(\frac{l(w)}{2})$. Since f is nondecreasing, the result follows. $\square$

In [4], Bridson and Riley give an example of a finitely presented group G whose (minimal) intrinsic and extrinsic diameter functions are not Lipschitz equivalent. While we have not resolved the relationship between tame filling inequalities in general, we give bounds on their interconnections in Theorem 2.2. These relationships between the intrinsic and extrinsic tame filling inequalities are applied in examples later in this paper.

Theorem 2.2. *Let G be a finitely presented group with Cayley complex X and combed filling $\mathcal{D}$. Suppose that $j : \mathbb{N} \to \mathbb{N}$ is a nondecreasing function such that for every vertex v of a van Kampen diagram Δ in $\mathcal{D}$, $d_\Delta(*, v) \le j(d_X(\epsilon, \pi_\Delta(v)))$, and let $\widetilde{j} : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be defined by $\widetilde{j}(n) := j(\lceil n \rceil) + 1$.*

(1) *If G satisfies an extrinsic tame filling inequality for the function f with respect to $\mathcal{D}$, then G satisfies an intrinsic tame filling inequality for the function $\widetilde{j} \circ f$.*
(2) *If G satisfies an intrinsic tame filling inequality for the function f with respect to $\mathcal{D}$, then G satisfies an extrinsic tame filling inequality for the function $f \circ \widetilde{j}$.*

Proof. We begin by showing that the inequality restriction for j on vertices holds for the function $\widetilde{j}$ on all points in the van Kampen diagrams in $\mathcal{D}$, using the fact that coarse distances on edges and 2-cells are closely linked to those of vertices. Let $(\Delta, \Psi) \in \mathcal{D}$ and let p be any point in Δ. Among the vertices in the boundary of the open cell of Δ containing p, let v be the vertex whose coarse distance to the basepoint $*$ is maximal. Then $\widetilde{d}_\Delta(*, p) \le \widetilde{d}_\Delta(*, v) + 1$. Moreover, $\pi_\Delta(v)$ is again a vertex in the boundary of the open cell of X containing $\pi_\Delta(p)$, and so $d_X(\epsilon, \pi_\Delta(v)) \le \lceil \widetilde{d}_X(\epsilon, \pi_\Delta(p)) \rceil$. Applying the fact that j is nondecreasing, then $\widetilde{d}_\Delta(*, p) \le \widetilde{d}_\Delta(*, v) + 1 \le j(\widetilde{d}_X(\epsilon, \pi_\Delta(v))) + 1 \le j(\lceil \widetilde{d}_X(\epsilon, \pi_{\Delta_w}(p)) \rceil) + 1$. Hence the second inequality in

$$\widetilde{d}_X(\epsilon, \pi_\Delta(p)) \le \widetilde{d}_\Delta(*, p) \le \widetilde{j}(\widetilde{d}_X(\epsilon, \pi_\Delta(p))) \qquad (a)$$

follows. The first inequality is a consequence of the fact that coarse distance can only be preserved or decreased by the natural map π_Δ from any van Kampen diagram to the Cayley complex.

Now suppose that G (with its finite presentation) satisfies an extrinsic tame filling inequality for the function $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ with respect to $\mathcal{D}$. Then for all $p \in \Delta$ and for all $0 \le s < t \le 1$, we have $\widetilde{d}_\Delta(*, \Psi(p, s)) \le \widetilde{j}(\widetilde{d}_X(\epsilon, \pi_\Delta(\Psi(p, s)))) \le \widetilde{j}(f(\widetilde{d}_X(\epsilon, \pi_\Delta(\Psi(p, t))))) \le \widetilde{j}(f(\widetilde{d}_\Delta(*, \Psi(p, t))))$ where the first and third inequalities follow from (a), and the second uses the extrinsic tame filling inequality and the nondecreasing property of $\widetilde{j}$. This completes the proof of (1).

The proof of (2) is similar. $\square$

3. Alternative views for tame filling inequalities

The definitions of intrinsic and extrinsic tame filling inequalities for finitely presented groups require a van Kampen diagram for each word over the generators that represents the trivial element of the group; i.e., a filling. In our first alternative view, we show that the required collection of diagrams can be reduced to a normal filling, at the cost of altering

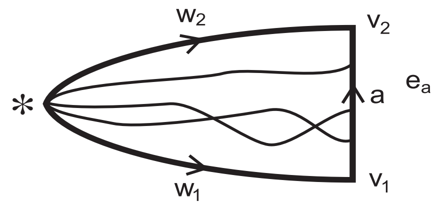

FIGURE 5. Edge homotopy

the direction of the homotopy paths. We will apply this view in Section 4 in developing an algorithm to bound tame filling inequalities for stackable groups.

In Definition 1.14, our definition of a van Kampen homotopy $\Psi : \partial\Delta \times [0,1] \to \Delta$ is "natural", in the sense that the first factor in the domain of this function is a subcomplex of the associated van Kampen diagram Δ. This requires that for each point p on an edge e of $\partial\Delta$, there is a unique choice of path from the basepoint $*$ to p via this homotopy. However, when traveling along the boundary word w counterclockwise around Δ, this point p (and undirected edge e) may be traversed more than once, and it can be convenient to have different combings of this edge corresponding to the different traversals. In this section, we also show that a tame filling inequality with respect to this more relaxed condition is equivalent to the tame filling inequality defined in Section 1.4. This second alternative view of the invariants will prove useful in Section 7, in our proof of the quasi-isometry invariance of tame filling inequalities.

We begin with definitions to make both of these statements more precise. First we extend the notion of tameness to homotopies with other domains.

Definition 3.1. *Let $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be a nondecreasing function, let Z be a space, and let $\alpha : Z \times [0,1] \to \Delta$ be a continuous function onto a van Kampen diagram Δ with basepoint $*$, satisfying $\alpha(p,0) = *$ for all $p \in Z$. The map α is called* intrinsically f-tame *if for all $p \in Z$ and $0 \leq s < t \leq 1$, we have*

$$\widetilde{d}_\Delta(*, \alpha(p,s)) \leq f(\widetilde{d}_\Delta(*, \alpha(p,t))) \ .$$

Similarly, the map α is extrinsically f-tame *if for all $p \in Z$ and $0 \leq s < t \leq 1$, we have*

$$\widetilde{d}_X(\epsilon, \pi_\Delta(\alpha(p,s))) \leq f(\widetilde{d}_X(\epsilon, \pi_\Delta(\alpha(p,t)))) \ .$$

Next we reroute the homotopy paths in a diagram Δ, so that all paths travel from the basepoint $*$ to a single edge of $\partial\Delta$. Suppose that Δ is a van Kampen diagram for a word $w = w_1 a w_2^{-1}$ such that each w_i is a word over A and $a \in A$. An *edge homotopy* in Δ of the directed edge e_a in $\partial\Delta$, from vertex v_1 to vertex v_2, corresponding to a is a continuous function $\Theta : e_a \times [0,1] \to \Delta$ satisfying

(e1): whenever p is a point in e_a, then $\Theta(p,0) = *$ and $\Theta(p,1) = p$,
(e2): for $i = 1,2$ the path $\Theta(v_i, \cdot) : [0,1] \to \Delta$ follows the path labeled w_i in $\partial\Delta$.

See Figure 5.

A *combed normal filling* for a group G with symmetrized presentation $\mathcal{P} = \langle A \mid R\rangle$ consists of a collection $\mathcal{N} \subseteq A^*$ of simple word normal forms for G together with a collection $\mathcal{E} = \{(\Delta_e, \Theta_e) \mid e \in E(X)\}$, where $E(X)$ is the set of edges in the Cayley complex X, satisfying the following:

(n1): For each edge e, there is a choice of direction for e so that if the initial vertex is g, the terminal vertex is h, the label on e in the Cayley graph is a, and y_g,y_h are the representatives of g,h in $\mathcal{N}$, then Δ_e is a van Kampen diagram for the word $w_e := y_g a y_h^{-1}$.

(n2): For each e, the map $\Theta_e : \hat{e} \times [0,1] \to \Delta_e$ is an edge homotopy of the directed edge $\hat{e}$, from vertex $\hat{g}_e$ to vertex $\hat{h}_e$, in $\partial\Delta_e$ that corresponds to a in the factorization of the boundary word w_e.

(n3): For every pair of edges $e, e' \in E(X)$ with a common endpoint g, we require that $\pi_{\Delta_e} \circ \Theta_e(\hat{g}_e, t) = \pi_{\Delta_{e'}} \circ \Theta_{e'}(\hat{g}_{e'}, t)$ for all t in $[0,1]$; that is, Θ_e and $\Theta_{e'}$ project to the same path, with the same parametrization, in the Cayley complex X.

Note that, as in the case of fillings, every combed normal filling induces a combed filling, again using the "seashell" method, as follows. Given a combed normal filling $(\mathcal{N}, \mathcal{E})$ and a word $w = a_1 \cdots a_n$ with $w =_G \epsilon$ and each $a_i \in A$, let (Δ_i, Θ_i) be the element of $\mathcal{E}$ corresponding to the edge of X with endpoints $a_1 \cdots a_{i-1}$ and $a_1 \cdots a_i$. If necessary replacing Δ_i by its mirror image and altering Θ_i accordingly, we may assume that Δ_i has boundary label $y_{i-1} a_i y_i$, where y_i is the normal form in $\mathcal{N}$ of $a_1 \cdots a_i$. As usual, let Δ_w be the van Kampen diagram for w obtained by successively gluing these diagrams along their y_i boundary subpaths. This procedure yields a quotient map $\alpha : \coprod \Delta_i \to \Delta_w$, such that each restriction $\alpha| : \Delta_i \to \Delta_w$ is an embedding. Let $\hat{e}_i$ be the edge in the boundary path of Δ_i (and by slight abuse of notation also in the boundary of Δ_w) corresponding to the letter a_i. In order to build a van Kampen homotopy on Δ_w, we note that the edge homotopies Θ_i give a continuous function $\alpha \circ \coprod \Theta_i : \coprod \hat{e}_i \times [0,1] \to \Delta_w$. Recall that property (n3) of the definition of combed normal filling says that on the common endpoint v_i of the edges $\hat{e}_i$ and $\hat{e}_{i+1}$ of Δ_w, the paths $\pi_{\Delta_i} \circ \Theta_i(v_i, \cdot)$ and $\pi_{\Delta_{i+1}} \circ \Theta_{i+1}(v_i, \cdot)$ follow the edge path in X labeled y_i with the same parametrization. Hence the same is true for the functions $\Theta_i(v_i, \cdot)$ and $\Theta_{i+1}(v_{i+1}, \cdot)$ following the edge paths labeled y_i that were glued by α. Moreover, if an $\hat{e}_i$ edge and (the reverse of) an $\hat{e}_j$ edge are glued via α, the maps Θ_i and Θ_j have been chosen to be consistent. Hence the collection of maps Θ_i are consistent on points identified by the gluing map α, and we obtain an induced function $\Psi_w : \partial\Delta_w \times [0,1] \to \Delta_w$. The edge homotopy conditions of the Θ_i maps imply that the function Ψ_w satisfies all of the properties needed for the required van Kampen homotopy on the diagram Δ_w.

Definition 3.2. *A group G with finite presentation $\langle A \mid R\rangle$ satisfies an* intrinsic *[resp.* extrinsic*]* tame normal inequality *for a nondecreasing function $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ if there is a combed normal filling given by $\mathcal{N} \subseteq A^*$, and $\mathcal{E} = \{(\Delta_u, \Theta_u) \mid u \in E(X)\}$ such that each edge homotopy Θ_u is intrinsically [resp. extrinsically] f-tame.*

A combed normal filling is *geodesic* if all of the words in the normal form set $\mathcal{N}$ label geodesics in the associated Cayley graph. We call a tame normal inequality *geodesic* if the associated combed normal filling is geodesic. A set of geodesic normal forms that we will

use several times in this paper is the set of *shortlex* normal forms. Choose a (lexicographic) total ordering on the finite set A. For any two words z, z' over A, we write $z <_{sl} z'$ if z is less than z' in the corresponding shortlex ordering on A^*.

Next we turn to relaxing the boundary condition. Let S^1 denote the unit circle in the $\mathbb{R}^2$ plane. For any natural number n, let C_n be S^1 with a 1-complex structure consisting of n vertices (one of which is the basepoint $(-1,0)$) and n edges.

Given any van Kampen diagram Δ over $\mathcal{P}$ for a word w of length n, let $\vartheta_\Delta : C_n \to \partial\Delta$ be the function that maps $(-1,0)$ to $*$ and, going counterclockwise once around C_n, maps each subsequent edge of C_n homeomorphically onto the next edge in the counterclockwise path labeled w along the boundary of Δ.

A *disk homotopy* of a van Kampen diagram Δ over $\mathcal{P}$ for a word w of length n is a continuous function $\Phi : C_n \times [0,1] \to \Delta$ satisfying:

- **(d1):** whenever $p \in C_n$, then $\Phi(p,0) = *$ and $\Phi(p,1) = \vartheta_\Delta(p)$,
- **(d2):** whenever $t \in [0,1]$, then $\Phi((-1,0),t) = *$, and
- **(d3):** whenever $p \in C_n^0$, then $\Phi(p,t) \in \Delta^1$ for all $t \in [0,1]$.

Definition 3.3. *A group G with finite presentation $\mathcal{P}$ satisfies an* intrinsic *[respectively,* extrinsic*]* relaxed tame filling inequality *for a nondecreasing function $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ if for all $w \in A^*$ with $w =_G \epsilon$, there exists a van Kampen diagram Δ for w over $\mathcal{P}$ and an intrinsically [resp. extrinsically] f-tame disk homotopy $\Phi : C_{l(w)} \times [0,1] \to \Delta$.*

Now we are ready to show that in the intrinsic case, these concepts are effectively equivalent.

Proposition 3.4. *Let G be a group with a finite symmetrized presentation $\mathcal{P}$, and let $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be a nondecreasing function. The following are equivalent, up to Lipschitz equivalence of the function f:*

1. *$(G,\mathcal{P})$ satisfies an intrinsic tame filling inequality with respect to f.*
2. *$(G,\mathcal{P})$ satisfies an intrinsic relaxed tame filling inequality with respect to f.*
3. *$(G,\mathcal{P})$ satisfies an intrinsic geodesic tame normal inequality with respect to f.*

Proof. Write the presentation for G as $\mathcal{P} = \langle A \mid R\rangle$.

(1) $\Rightarrow$ (2):

Given a van Kampen diagram Δ for a word w and a van Kampen homotopy $\Psi : \partial\Delta \times [0,1] \to \Delta$, the composition $\Phi = \Psi \circ (\vartheta_\Delta \times id_{[0,1]}) : C_n \times [0,1] \to \Delta$ is a disk homotopy for this diagram. The fact that the identity function is used on the [0,1] factor implies the result on the inequalities.

(2) $\Rightarrow$ (3):

Suppose that $\mathcal{D} = \{(\Delta_w, \Phi_w) \mid w \in A^*, w =_G \epsilon\}$ is a collection of van Kampen diagrams and disk homotopies such that each Φ_w is intrinsically f-tame.

Let $\mathcal{N} := \{y_g \mid g \in G\}$ be the set of shortlex normal forms with respect to some lexicographic ordering of A.

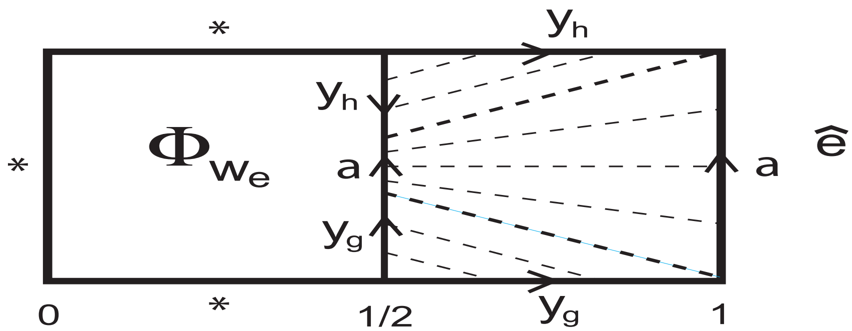

FIGURE 6. Edge homotopy Θ_e in Proposition 3.4 proof (of (2) $\Rightarrow$ (3))

For any edge $e \in E(X)$, we orient the edge e from vertex g to $h = ga$ if $y_g <_{sl} y_{ga}$. There is a pair $(\Delta_{w_e}, \Phi_{w_e})$ in $\mathcal{D}$ associated to the word $w_e := y_g a y_h^{-1}$. Define $\Delta_e := \Delta_{w_e}$, and let $\hat{e}$ be the edge in the boundary path of Δ_{w_e} corresponding to the letter a in the concatenated word w_e.

We construct an edge homotopy $\Theta_e : \hat{e} \times [0,1] \to \Delta_e$ as follows. Associated with the map Φ_{w_e} we have a canonical map $\vartheta_{\Delta_e} : C_{l(w_e)} \to \partial\Delta_e$, given by $\vartheta_{\Delta_e}(q) = \Phi_{w_e}(q,1)$, using disk homotopy condition (d1). Recall that this map wraps the simple edge circuit $C_{l(w_e)}$ cellularly along the edge path of $\partial\Delta_e$. Let $\gamma : \hat{e} \to C_{l(w_e)}$ be a continuous map that wraps the edge $\hat{e}$ once (at constant speed) in the counterclockwise direction along the circle, with the endpoints of $\hat{e}$ mapped to $(-1,0)$. For each point p in $\hat{e}$, and for all $t \in [0, \frac{1}{2}]$, define $\Theta_e(p,t) := \Phi_{w_e}(\gamma(p), 2t)$. Then $\Theta_e(p, \frac{1}{2}) = \vartheta_{\Delta_e}(\gamma(p)) \in \partial\Delta_e$.

Let $\widetilde{e}$ be the (directed) edge of $C_{l(w_e)}$ corresponding to the edge $\hat{e}$ of the boundary path in $\partial\Delta_e$, with endpoint v_1 of $\widetilde{e}$ occurring earlier than endpoint v_2 in the counterclockwise path from $(-1,0)$. Also let $\widetilde{r}_1$, $\widetilde{r}_2$ be the arcs of $C_{l(w_e)}$ mapping via ϑ_{Δ_e} to the paths labeled by the subwords y_g, y_h^{-1}, respectively, of w_e in $\partial\Delta_e$. For each point p in the interior $Int(\hat{e})$ of the edge $\hat{e}$, there is a unique point $\widetilde{p}$ in $\widetilde{e}$ with $\vartheta_{\Delta_e}(\widetilde{p}) = p$. There is an arc (possibly a single point) in $C_{l(w_e)}$ from $\gamma(p)$ to $\widetilde{p}$ that is disjoint from the point $(-1,0)$; let $\delta_p : [\frac{1}{2}, 1] \to C_{l(w_e)}$ be the constant speed path following this arc. That is, $\vartheta_{\Delta_e} \circ \delta_p$ is a path in $\partial\Delta_e$ from $\vartheta_{\Delta_e}(\gamma(p))$ to p. In particular, if $\gamma(p)$ lies in $\widetilde{r}_1$, then the path $\vartheta_{\Delta_e} \circ \delta_p$ follows the end portion of the boundary path labeled by y_g from $\vartheta_{\Delta_e}(\gamma(p))$ to the endpoint $\vartheta_{\Delta_e}(v_1)$ of $\hat{e}$ and then follows a portion of $\hat{e}$ to p. If $\gamma(p)$ lies in $\widetilde{r}_2$, the path $\vartheta_{\Delta_e} \circ \delta_p$ follows a portion of the boundary path y_h and $\hat{e}$ clockwise from $\vartheta_{\Delta_e} \circ \delta_p$ via $\vartheta_{\Delta_e}(v_2)$ to p, and if $\gamma(p)$ is in $\widetilde{e}$, then the path $\vartheta_{\Delta_e} \circ \delta_p$ remains in $\hat{e}$. Finally, for each point p that is an endpoint $p = \vartheta_{\Delta_e}(v_i)$ (with $i = 1,2$), let $\delta_p : [\frac{1}{2}, 1] \to C_{l(w_e)}$ be the constant speed path along the arc $\widetilde{r}_i$ in $C_{l(w_e)}$ from $(-1,0)$ to v_i. Now for all p in $\hat{e}$ and $t \in [\frac{1}{2}, 1]$, define $\Theta_e(p,t) := \vartheta_{\Delta_e}(\delta_p(t))$.

Combining the last sentences of the previous two paragraphs, we have constructed a continuous function $\Theta_e : \hat{e} \times [0,1] \to \Delta_e$. See Figure 6 for an illustration of this map. The disk homotopy conditions satisfied by Φ_{w_e} imply that Θ_e is an edge homotopy. Let $\mathcal{E}$ be the

collection $\mathcal{E} = \{(\Delta_e, \Theta_e) \mid e \in E(X)\}$. Then $\mathcal{N}$ together with $\mathcal{E}$ define a geodesic combed normal filling of the pair $(G, \mathcal{P})$.

Now we turn to analyzing the tameness of the edge homotopy $\Theta_e : \hat{e} \times [0,1] \to \Delta_e$. Suppose that p is any point in $\hat{e}$. The fact that the disk homotopy Φ_{w_e} is intrinsically f-tame implies that for all $0 \le s < t \le \frac{1}{2}$, we have $\widetilde{d}_{\Delta_e}(*, \Theta_e(p,s)) \le f(\widetilde{d}_{\Delta_e}(*, \Theta_e(p,t)))$.

The path $\Theta_e(p, \cdot) = \vartheta_{\Delta_e}(\delta_p(\cdot)) : [\frac{1}{2}, 1] \to \Delta_e$ on the second half of the interval $[0,1]$ follows a geodesic in Δ_e, with the possible exception of the end portion of this path that lies completely contained in the edge $\hat{e}$. Hence for all $\frac{1}{2} \le s < t \le 1$, we have $\widetilde{d}_{\Delta_e}(*, \Theta_e(p,s)) \le \widetilde{d}_{\Delta_e}(*, \Theta_e(p,t)) + 1$.

Finally, whenever $0 \le s < \frac{1}{2} < t \le 1$, we have $\widetilde{d}_{\Delta_e}(*, \Theta_e(p,s)) \le f(\widetilde{d}_{\Delta_e}(*, \Theta_e(p,\frac{1}{2}))) \le f(\widetilde{d}_{\Delta_e}(*, \Theta_e(p,t)) + 1)$, where the latter inequality uses the nondecreasing property of f.

Putting these three cases together, the edge homotopy Θ_e is intrinsically g-tame with respect to the nondecreasing function $g : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ given by $g(n) = f(n+1)$ for all $n \in \mathbb{N}[\frac{1}{4}]$, and this function is Lipschitz equivalent to f.

(3) $\Rightarrow$ (1):

Suppose that the set $\mathcal{N} = \{y_g \mid g \in G\}$ of geodesic normal forms for G together with the collection $\mathcal{E} = \{(\Delta_e, \Theta_e) \mid e \in E(X)\}$ is a combed normal filling with each Θ_e intrinsically f-tame. Let $\mathcal{D} = \{(\Delta'_w, \Psi_w)\}$ be the induced combed filling from the seashell method.

Fix a word $w \in A^*$ with $w =_G \epsilon$. Recall that there is a quotient map $\alpha : \coprod \Delta_i \to \Delta'_w$ from the seashell construction, where each Δ_i is a normal form diagram from $\mathcal{E}$, and the restriction of α to each Δ_i is an embedding. In particular, for each i, paths labeled $y_i \in \mathcal{N}$ in $\partial\Delta_i$ and $\partial\Delta_{i+1}$ are glued via α. Moreover, the homotopy Ψ_w is induced by the map $\alpha \circ \coprod \Theta_i : \coprod \hat{e}_i \times [0,1] \to \Delta'_w$.

In order to analyze coarse distances in the van Kampen diagram Δ'_w, we begin by supposing that p is any vertex in Δ'_w. Then $p = \alpha(q)$ for some vertex $q \in \Delta_i$ (for some i). The identification map α cannot increase distances to the basepoint, so we have $d_{\Delta'_w}(*, p) \le d_{\Delta_i}(*, q)$. Suppose that $\beta : [0,1] \to \Delta'_w$ is an edge path in Δ'_w from $\beta(0) = *$ to $\beta(1) = p$ of length strictly less than $d_{\Delta_i}(*, q)$. This path cannot stay in the (closed) subcomplex $\alpha(\Delta_i)$ of Δ'_w, and so there is a minimum time $0 < s \le 1$ such that $\beta(t) \in \alpha(\Delta_i)$ for all $t \in [s, 1]$. Then the point $\beta(s)$ must lie on the image of the boundary of Δ_i in Δ'_w. Since the words y_{i-1} and y_i label geodesics in the Cayley complex of the presentation $\mathcal{P}$, these words must also label geodesics in Δ_i and Δ'_w. Hence we can replace the portion of the path β on the interval $[0, s]$ with the geodesic path along one of these words from $*$ to $\beta(s)$, to obtain a new edge path in $\alpha(\Delta_i)$ from $*$ to p of length strictly less than $d_{\Delta_i}(*, q)$. Since α embeds Δ_i in Δ'_w, this results in a contradiction. Thus for each vertex p in Δ'_w, we have $d_{\Delta'_w}(*, p) = d_{\Delta_i}(*, q)$.

For any point in the interior of an edge or 2-cell in Δ'_w, the coarse distance in Δ'_w to the basepoint is computed from the path metric distances of the vertices in the boundary of the cell. Hence the result of the previous paragraph shows that for any point p in Δ'_w with $p = \alpha(q)$ for some point $q \in \Delta_i$, we have $\widetilde{d}_{\Delta'_w}(*, p) = \widetilde{d}_{\Delta_i}(*, q)$. That is, the map α preserves coarse distance.

Finally, for each $p \in \partial\Delta'_w$, we have $p = \alpha(q)$ for some point q in $\hat{e}_i \subseteq \Delta_i$, and so $\Psi_w(p,t) = \alpha(\Theta_i(q,t))$ for all $t \in [0,1]$. Since each Θ_i is an intrinsically f-tame map, the homotopy Ψ_w is also f-tame. □

In the extrinsic setting, a combed normal filling gives rise to another type of homotopy, namely the notion of a 1-combing, first defined by Mihalik and Tschantz in [18]. A *1-combing* of the Cayley complex X is a continuous function $\Upsilon : X^1 \times [0,1] \to X$ satisfying that whenever $p \in X^1$, then $\Upsilon(p,0) = \epsilon$ and $\Upsilon(p,1) = p$, and whenever $p \in X^0$, then $\Upsilon(p,t) \in X^1$ for all $t \in [0,1]$. That is, a 1-combing is a continuous choice of paths in the Cayley 2-complex X from the vertex ϵ labeled by the identity of G to each point of the Cayley graph X^1, such that the paths to vertices are required to stay inside the 1-skeleton.

For a combed normal filling given by a set $\mathcal{N}$ of normal forms together with a collection $\mathcal{E} = \{(\Delta_u, \Theta_u) \mid u \in E(X)\}$ of van Kampen diagrams and edge homotopies, there is a canonical associated 1-combing Υ of the Cayley complex X defined as follows. For any point p in the Cayley graph X^1, let u be an edge in X containing p. Then for any $t \in [0,1]$, define $\Upsilon(p,t) := \pi_{\Delta_u}(\Theta_u(p,t))$. The consistency condition (n3) of the definition of a combed normal filling ensures that Υ is well-defined.

In fact, this associated 1-combing satisfies more restrictions than those of Mihalik and Tschantz, in that the 1-combing factors through edge homotopies of normal form diagrams. We refer to these extra properties as diagrammatic. That is, a *diagrammatic 1-combing* of X is a 1-combing $\Upsilon : X^1 \times [0,1] \to X$ that also satisfies:

- **(c1):** whenever $t \in [0,1]$, then $\Upsilon(\epsilon,t) = \epsilon$.
- **(c2):** whenever v is a vertex in X, the path $\Upsilon(v,\cdot)$ follows an embedded edge path (i.e., no repeated vertices or edges) from ϵ to v labeled by word w_v, and
- **(c3):** whenever e is a directed edge from vertex u to vertex v in X labeled by a, then there is a van Kampen diagram Δ with respect to $\mathcal{P}$ for the word $w_u a w_v^{-1}$, together with an edge homotopy $\Theta : \hat{e} \times [0,1] \to \Delta$ associated to the edge $\hat{e}$ of $\partial\Delta$ corresponding to the letter a in this boundary word, such that $\Upsilon \circ (\pi_\Delta \times id_{[0,1]})|_{\hat{e}\times[0,1]} = \pi_\Delta \circ \Theta$.

Mihalik and Tschantz [18] defined a notion of tameness of a 1-combing, which Hermiller and Meier [12] refined to the idea of the 1-combing homotopy being f-tame with respect to a function f, which they call a "radial tameness function". (In [12], coarse distance in X is described in terms of "levels", and the definition of coarse distance for a 2-cell is defined slightly differently from that of Definition 1.15.)

Definition 3.5. [12] *A group G with finite presentation $\mathcal{P}$ satisfies a* radial tame combing inequality *for a nondecreasing function $\rho : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ if there is a diagrammatic 1-combing Υ of the associated Cayley 2-complex X such that*

($\dagger^r$)**:** *for all $p \in X^1$ and $0 \le s < t \le 1$, we have*

$$\widetilde{d}_X(\epsilon, \Upsilon(p,s)) \le \rho(\widetilde{d}_X(\epsilon, \Upsilon(p,t))) \ .$$

Note that the radial tame combing inequality property is fundamentally an extrinsic property, utilizing (coarse) distances measured in the Cayley complex. Effectively, Proposition 3.6 below shows that the logical intrinsic analog of a radial tame combing inequality is the concept of an intrinsic tame filling inequality or intrinsic tame normal inequality.

In Proposition 3.6, we show that in the extrinsic setting, a stronger set of equivalences hold. In particular, not only are the intrinsic properties discussed in the Proposition 3.4 also equivalent in the extrinsic case, they are also equivalent to a radial tame combing inequality, and to an extrinsic tame normal inequality without the geodesic restriction.

Proposition 3.6. *Let G be a group with a finite symmetrized presentation $\mathcal{P}$, and let $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be a nondecreasing function. The following are equivalent, up to Lipschitz equivalence of the function f:*

(1) *$(G, \mathcal{P})$ satisfies an extrinsic tame filling inequality with respect to f.*
(2) *$(G, \mathcal{P})$ satisfies an extrinsic relaxed tame filling inequality with respect to f.*
(3) *$(G, \mathcal{P})$ satisfies an extrinsic geodesic tame normal inequality with respect to f.*
(4) *$(G, \mathcal{P})$ satisfies an extrinsic tame normal inequality with respect to f.*
(5) *$(G, \mathcal{P})$ satisfies a radial tame combing inequality with respect to f.*

Proof. We first note that the proofs of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are analogous to the proofs of the same intrinsic properties in Proposition 3.4. The implication (3) $\Rightarrow$ (4) is immediate.

The implication (4) $\Rightarrow$ (1) follows the seashell method as in the proof of (3) $\Rightarrow$ (1) in Proposition 3.4. In the extrinsic setting, the seashell quotient map α preserves extrinsic distances (irrespective of whether or not the normal forms are geodesics). That is, for any point p in Δ_i, we have $\pi_{\Delta_i}(p) = \pi_{\Delta'_w}(p)$, and hence $\widetilde{d}_X(\epsilon, \pi_{\Delta_i}(p)) = \widetilde{d}_X(\epsilon, \pi_{\Delta'_w}(p))$. The rest of the proof follows.

The implication (4) $\Rightarrow$ (5) utilizes the canonical diagrammatic 1-combing Υ associated to a combed normal filling discussed above. For f-tame edge homotopies Θ_e in the combed normal filling, the definition $\Upsilon(p,t) := \pi_{\Delta_e}(\Theta_e(p,t))$, implies that the condition $(\dagger^r)$ (with respect to the same function f) also holds.

Finally, for the implication (5) $\Rightarrow$ (4), given a diagrammatic 1-combing Υ, the definition of diagrammatic implies that there is a canonically associated combed normal filling through which Υ factors, as well. Again we have $(\dagger^r)$ implies that each of the edge homotopies is f-tame, with respect to the same function f, as an immediate consequence. □

We note that each of the properties in Propositions 3.4 and 3.6 must also have the same quasi-isometry invariance as the respective tame filling inequality, from Theorem 7.1.

4. Combed fillings for stackable groups

In this section we give an inductive procedure for constructing a combed normal filling for any stackable group. Recall that in Section 1.3, an inductive procedure was described for building a normal filling from a stacking; in this section we extend this process to include edge homotopies. Because edge homotopies are built in this recursive fashion, we will have finer control on their tameness than for a more general combed normal filling. We will

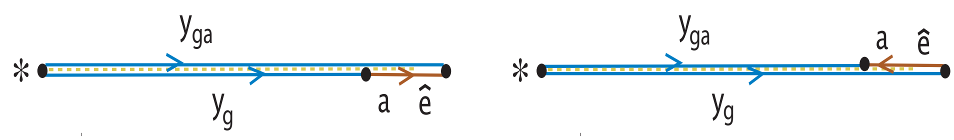

FIGURE 7. (Δ_e, Θ_e) for e in $\vec{E}_d$

utilize this extra restriction to prove in Theorem 4.2 that every stackable group admits finite-valued intrinsic and extrinsic tame filling inequalities.

Let G be a stackable group with stacking $(\mathcal{N}, c)$ over a finite inverse-closed generating set A, and let $\mathcal{P} = \langle A \mid R_c \rangle$ be the (symmetrized) stacking presentation, with Cayley complex X. Let $\vec{E}_d$ be the set of degenerate edges in X, let $\vec{E}_r$ be the set of recursive edges, and let $<_c$ be the stacking ordering. We construct a combed normal filling for G as follows.

The set $\mathcal{N}$ will also be the set of normal forms for the combed normal filling. For each $g \in G$, let y_g denote the normal form for g in $\mathcal{N}$.

For each directed edge e in $\vec{E}(X) = \vec{E}_d \cup \vec{E}_r$, oriented from a vertex g to a vertex h and labeled by $a \in A$, let $w_e := y_g a y_h^{-1}$. Let Δ_e be the normal form diagram (with boundary word w_e) associated to e, obtained from the stacking by using the construction in Section 1.3.

In the case that e lies in $\vec{E}_d$, the diagram Δ_e contains no 2-cells. Let $\hat{e}$ be the edge of $\partial\Delta_e$ corresponding to a in the factorization of w_e; see Figure 7. Define the edge homotopy $\Theta_e : \hat{e} \times [0,1] \to \Delta_e$ by taking $\Theta_e(p, \cdot) : [0,1] \to \Delta_e$ to follow the shortest length (i.e. geodesic with respect to the path metric) path from the basepoint $*$ to p at a constant speed, for each point p in $\hat{e}$.

Next we use the recursive construction of the van Kampen diagram Δ_e to recursively construct the edge homotopy in the case that $e \in \vec{E}_r$. Recall that if we write $c(e) = a_1 \cdots a_n$ with each $a_i \in A^*$, then the normal form diagram Δ_e is constructed from normal form diagrams Δ_i with boundary labels $y_{ga_1 \cdots a_{i-1}} a_i y_{ga_1 \cdots a_i}$, obtained by induction or from degenerate edges. These diagrams are glued along their common boundary paths $y_i := y_{ga_1 \cdots a_i}$ (to obtain the "seashell" diagram Δ_e'), and then a single 2-cell with boundary label $c(e)a^{-1}$ is glued onto Δ_e' along the $c(e)$ subpath of $\partial\Delta_e'$, to produce Δ_e.

A slightly alternative view of this construction of Δ_e will allow us more flexibility in constructing the edge homotopy associated to this diagram, which in turn will lead to better tameness bounds later. Factor $c(e) = x_g^{-1} \tilde{c}_e x_h$ such that the directed edges in the paths in X labeled by x_g^{-1} starting at g, and labeled by x_h^{-1} starting at h, all lie in $\vec{E}_d$, and such that $y_g = y_q x_g$ and $y_h = y_r x_h$ where $q :=_G g x_g^{-1}$ and $r :=_G h x_h^{-1}$. There are indices j, k such that $\tilde{c}_e := a_j \cdots a_k$. If the word $\tilde{c}_e$ is nonempty, then Δ_e can also be constructed by a seashell gluing of the normal form diagrams $\Delta_j, \cdots, \Delta_k$ to produce a diagram Δ_e'' with boundary labeled $y_q \tilde{c}_e y_r^{-1}$, after which a single 2-cell f_e with boundary label $c(e)a^{-1}$ is glued onto Δ_e'', along the $\tilde{c}_e$ subpath in $\partial\Delta_e''$, to produce Δ_e. If the word $\tilde{c}_e$ is empty, then $q = r$, and Δ_e is obtained by taking a simple edge path from a basepoint labeled by the word y_q (i.e., the van Kampen diagram for the word $y_q y_q^{-1}$ with no 2-cells), and attaching

a single 2-cell f_e with boundary label $c(e)a^{-1}$, gluing the end of the y_q edge path to the vertex of ∂f_e separating the x_g^{-1} and x_h subpaths. It follows from this construction that the diagrams Δ_i and the cell f_e can be considered to be subsets of Δ_e.

Let $\hat{e}$ be the directed edge in $\partial\Delta_e$ from vertex $\hat{g}$ to vertex $\hat{h}$ corresponding to a in the factorization of w_e. Let $\hat{q}$ and $\hat{r}$ be the vertices of the 2-cell f_e at the start and end, respectively, of the path in ∂f_e labeled by $\tilde{c}_e$. Let $J : \hat{e} \to [0,1]$ be a homeomorphism, with $J(\hat{g}) = 0$ and $J(\hat{h}) = 1$. Since f_e is a disk, there is a continuous function $\Xi_e : \hat{e} \times [0,1] \to f_e$ such that: (i) For each p in the interior $Int(\hat{e})$, we have $\Xi_e(p,(0,1)) \subseteq Int(f_e)$ and $\Xi_e(p,1) = p$. (ii) $\Xi_e(J^{-1}(\cdot),0) : [0,1] \to f_e$ follows the path in ∂f_e labeled $\tilde{c}_e$ from $\hat{q}$ to $\hat{r}$ at constant speed. (iii) $\Xi_e(\hat{g},\cdot)$ follows the path in ∂f_e labeled x_g from $\hat{q}$ to $\hat{g}$ at constant speed. (iv) $\Xi_e(\hat{h},\cdot)$ follows the path in ∂f_e labeled x_h from $\hat{r}$ to $\hat{h}$ at constant speed. Let l_g, m_g, l_h, and m_h be the lengths of the words y_q, x_g, y_r, and x_h in A^*, respectively.

We give a piecewise definition of the edge homotopy $\Theta_e : \hat{e} \times [0,1] \to \Delta_e$ as follows. For any point p in $\hat{e}$, if $\tilde{c}_e$ is a nonempty word, then there is an index $j \le i \le k$ such that the point $\Xi_e(p,0)$ lies in Δ_i. In the case that $\tilde{c}_e = 1$, let $\Theta_i(\hat{q},\cdot)$ in the following formula denote the constant speed path following the geodesic in Δ_e from $*$ to $\hat{q} = \hat{r}$. Define

$$\Theta_e(p,t) := \begin{cases} \Theta_i(\Xi_e(p,0), \frac{1}{a_p}t) & \text{if } t \in [0,a_p] \\ \Xi_e(p, \frac{1}{1-a_p}(t-a_p)) & \text{if } t \in [a_p,1] \end{cases}$$

where

$$a_p := \begin{cases} \frac{2l_g}{l_g+m_g}(\frac{1}{2} - J(p)) + J(p) & \text{if } J(p) \in [0,\frac{1}{2}] \\ (1-J(p)) + \frac{2l_h}{l_h+m_h}(J(p) - \frac{1}{2}) & \text{if } J(p) \in [\frac{1}{2},1] \end{cases}$$

Note that if $a_p = 0$ and $J(p) \in [0,\frac{1}{2}]$, then we must also have $J(p) = 0$ and $l_g = 0$. In this case $p = \hat{g}$ and y_q is the empty word, and so $\Theta_i(\Xi_e(p,0),\cdot) = \Theta_i(\hat{q},\cdot)$ is a constant path at the basepoint $*$ of Δ_e; hence Θ_e is well-defined in this case. The other instances in which a_p can equal 0 or 1 are similar.

The complication in this definition of Θ_e stems from the need to ensure that for the endpoint vertices $\hat{g}$ and $\hat{h}$ of $\hat{e}$, the projections to X of the paths $\Theta_e(\hat{g},\cdot)$ and $\Theta_e(\hat{h},\cdot)$ via the map π_{Δ_e} are consistent with the paths defined for all other edges to these points; that is, to ensure that the property (n3) of the definition of combed normal filling will hold. In particular, we ensure that the paths $\Theta_e(\hat{g},\cdot)$, $\Theta_e(\hat{h},\cdot)$ follow the words y_g, y_h, respectively, in $\partial\Delta_e$ at constant speed. The van Kampen diagram Δ_e and edge homotopy Θ_e are illustrated in Figure 8.

We now have a collection of van Kampen diagrams and edge homotopies for the elements of $\vec{E}(X)$. To obtain the combed normal filling associated to the stacking, the final step again is to eliminate repetitions. Given any undirected edge e in $E(X)$, let $(\Delta_, \Theta_e)$ be a normal form diagram and edge homotopy constructed above for one of the orientations of e. Then the collection $\mathcal{N}$ of prefix-closed normal forms from the stacking, together with this collection $\mathcal{E} := \{(\Delta_e, \Theta_e) \mid e \in E(X)\}$ of van Kampen diagrams and edge homotopies, is a combed normal filling for G.

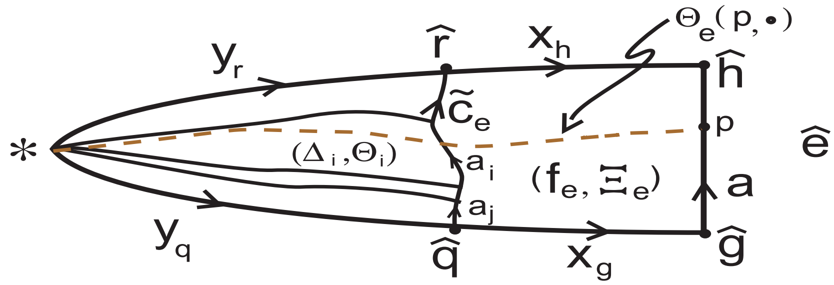

FIGURE 8. (Δ_e, Θ_e) for e in $\vec{E}_r$

Definition 4.1. *A* recursive combed normal filling *is a combed normal filling that can be constructed from a stacking by the above procedure. A* recursive combed filling *is a combed filling induced by a recursive combed normal filling using seashells.*

The extra structure of this recursively defined combed normal filling $(\mathcal{N}, \mathcal{E})$ allows us to compute finite-valued tame filling inequalities for G. To analyze the tameness of the edge homotopies, we consider the *intrinsic diameter* $idiam(\Delta_e)$ and *extrinsic diameter* $ediam(\Delta_e)$ of each van Kampen diagram in the collection $\mathcal{E}$; that is, $idiam(\Delta_e) = \max\{d_{\Delta_e}(\epsilon, v) \mid v \in \Delta_e^0\}$ and $ediam(\Delta_e) = \max\{d_X(\epsilon, \pi_{\Delta_e}(v)) \mid v \in \Delta_e^0\}$, where X is the Cayley complex of the stacking presentation. Let $B(n)$ be the ball of radius n (with respect to path metric distance) in the Cayley graph X^1 centered at ϵ. Define the functions $k_{\mathcal{N}}^i, k_{\mathcal{N}}^e, k_r^i, k_r^e : \mathbb{N} \to \mathbb{N}$ by

$$\begin{aligned} k_{\mathcal{N}}^i(n) &:= \max\{l(y_g) \mid g \in B(n)\}\ , \\ k_{\mathcal{N}}^e(n) &:= \max\{d_X(\epsilon, x) \mid \exists\, g \in B(n) \text{ such that } x \text{ is a prefix of } y_g\}\ , \\ k_r^i(n) &:= \max\{idiam(\Delta_e) \mid e \in \vec{E}_r \text{ and the initial vertex of } e \text{ is in } B(n)\}\ , \\ k_r^e(n) &:= \max\{ediam(\Delta_e) \mid e \in \vec{E}_r \text{ and the initial vertex of } e \text{ is in } B(n)\}\ . \end{aligned}$$

Note that we do not assume that prefixes are proper. (Also note that the van Kampen diagrams in the combed filling induced by the recursive combed normal filling $(\mathcal{N}, \mathcal{E})$ may not realize the minimal possible intrinsic or extrinsic diameter among all van Kampen diagrams for the same boundary words.)

We will need to consider coarse distances throughout the Cayley complex X. To that end, define the functions $\mu^i, \mu^e : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ by

$$\begin{aligned} \mu^i(n) &:= \max\{k_{\mathcal{N}}^i(\lceil n \rceil + 1) + 1, n+1, k_r^i(\lceil n \rceil + \zeta + 1)\} \text{ and} \\ \mu^e(n) &:= \max\{k_{\mathcal{N}}^e(\lceil n \rceil + 1) + 1, n+1, k_r^e(\lceil n \rceil + \zeta + 1)\}\ , \end{aligned}$$

where ζ is the length of the longest relator in the stacking presentation $\mathcal{P}$. It follows directly from the definitions that $k_{\mathcal{N}}^i, k_{\mathcal{N}}^e, k_r^i, k_r^e$ are nondecreasing functions, and therefore so are μ^i and μ^e.

Theorem 4.2. *If G is a stackable group, then G admits an intrinsic tame filling inequality for the finite-valued function μ^i, and an extrinsic tame filling inequality for the finite-valued function μ^e.*

Proof. Let $\mathcal{D} = \{(\Delta_w, \Psi_w)\}$ be the combed filling obtained via the seashell method from the recursive combed normal filling $(\mathcal{N}, \mathcal{E})$ associated to a stacking $(\mathcal{N}, c)$ for G. As usual, we write $\mathcal{N} = \{y_g \mid g \in G\}$. Let Δ_w be any of the van Kampen diagrams in $\mathcal{D}$, let p be any point in $\partial\Delta_w$, and let $0 \leq s < t \leq 1$. To simplify notation later, we also let $\sigma := \Psi_w(p, s)$ and $\tau := \Psi_w(p, t)$.

If τ is in the 1-skeleton Δ_w^1, then let $\tau' := \tau$ and $t' := t$. Otherwise, τ is in the interior of a 2-cell, and there is a $t \leq t' \leq 1$ such that $\Psi_w(p, [t, t'))$ is contained in that open 2-cell, and $\tau' := \Psi_w(p, t') \in \Delta_w^1$.

Case I. $\tau' \in \Delta_w^0$ is a vertex. In this case the path $\Psi_w(p, \cdot) : [0, t'] \to X$ follows the edge path labeled $y_{\pi_{\Delta_w}(\tau')}$ from $*$, through σ, to $\tau = \tau'$ (at constant speed). There is a vertex σ' on this path lying on the same edge as σ (with $\sigma' = \sigma$ if σ is a vertex) satisfying $\widetilde{d}_{\Delta_w}(*, \sigma) < d_{\Delta_w}(*, \sigma') + 1$ and $\widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\sigma)) < d_X(\epsilon, \pi_{\Delta_w}(\sigma')) + 1$. The subpath from $*$ to σ' is labeled by a prefix x of the word $y_{\pi_{\Delta_w}(\tau)}$. Then

$$\widetilde{d}_{\Delta_w}(*, \sigma) < d_{\Delta_w}(*, \sigma') + 1 \leq l(y_{\pi_{\Delta_w}(\tau)}) + 1 \leq k_{\mathcal{N}}^i(d_X(\epsilon, \pi_{\Delta_w}(\tau))) + 1 \leq k_{\mathcal{N}}^i(d_{\Delta_w}(*, \tau)) + 1$$

$$\text{and } \widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\sigma)) < d_X(\epsilon, \pi_{\Delta_w}(\sigma')) + 1 \leq k_{\mathcal{N}}^e(d_X(\epsilon, \pi_{\Delta_w}(\tau))) + 1 .$$

Case II. τ' is in the interior of an edge $\hat{e}$ of Δ_w. From the seashell construction, the path $\Psi_w(p, \cdot) : [0, 1] \to \Delta_w$ lies in a subdiagram Δ' of Δ_w such that Δ' is a normal form diagram in $\mathcal{E}$. From the construction of the recursive combed normal filling, the subpath $\Psi_w(p, \cdot) : [0, t'] \to \Delta_w$ lies in a subdiagram Δ_e of Δ' for some pair $(\Delta_e, \Theta_e) \in \mathcal{E}$ associated to a directed edge $e \in \vec{E}_d \cup \vec{E}_r$. Moreover, $\hat{e}$ is the edge of Δ_e corresponding to e, and the path $\Psi_w(p, \cdot) : [0, t'] \to \Delta_w$ is a bijective (orientation preserving) reparametrization of the path $\Theta_e(\tau', \cdot) : [0, 1] \to \Delta_e$.

Case IIA. $e \in \vec{E}_d$. The van Kampen diagram Δ_e contains no 2-cells, and the path $\Theta_e(\tau', \cdot) : [0, 1] \to \Delta_e$ follows the edge path labeled by a normal form $y_g \in \mathcal{N}$ from $*$ to $\hat{g}$ (at constant speed), and then follows the portion of $\hat{e}$ from $\hat{g}$ to τ', where $\hat{g}$ is the endpoint of $\hat{e}$ closest to $*$ in the diagram Δ_e. In this case, τ must also lie in Δ_w^1, and so again we have $\tau = \tau'$. Since $\hat{g}$ and τ lie in the same closed 1-cell, we have $d_{\Delta_w}(*, \hat{g}) < \lceil \widetilde{d}_{\Delta_w}(*, \tau) \rceil + 1$, and similarly for their images (via π_{Δ_w}) lying in the same closed edge of X.

If σ lies in the y_g path, then Case I applies to that path, with τ replaced by the vertex $\hat{g}$. Combining this with the inequality above and applying the nondecreasing property of the functions $k_{\mathcal{N}}^i$ and $k_{\mathcal{N}}^e$ yields

$$\widetilde{d}_{\Delta_w}(*, \sigma) < k_{\mathcal{N}}^i(d_{\Delta_w}(*, \hat{g})) + 1 \leq k_{\mathcal{N}}^i(\lceil \widetilde{d}_{\Delta_w}(*, \tau) \rceil + 1) + 1 \text{ and}$$

$$\widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\sigma)) < k_{\mathcal{N}}^e(d_X(\epsilon, \pi_{\Delta_w}(\hat{g}))) + 1 \leq k_{\mathcal{N}}^e(\lceil \widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\tau)) \rceil + 1) + 1 .$$

On the other hand, if σ lies in $\hat{e}$, then σ and τ are contained in a common edge. Hence

$$\widetilde{d}_{\Delta_w}(*, \sigma) < \widetilde{d}_{\Delta_w}(*, \tau) + 1 \text{ and } \widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\sigma)) < \widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\tau)) + 1 .$$

Case IIB. $e \in \vec{E}_r$. In this case either $\tau = \tau'$, or τ is in the interior of the cell f_e of the diagram Δ_e. Let g be the initial vertex of the directed edge e. Then g and $\pi_{\Delta_w}(\tau)$ lie in a common edge or 2-cell of X, and so $d_X(\epsilon, g) < \lceil \widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\tau)) \rceil + \zeta + 1$, where ζ is the length of the longest relator in the presentation of G.

Note that distances in the subdiagram Δ_e are bounded below by distances in Δ_w. In this case, combining these inequalities and the nondecreasing properties of k_r^i and k_r^e yields

$$\begin{aligned} \widetilde{d}_{\Delta_w}(*, \sigma) &\leq \widetilde{d}_{\Delta_e}(*, \sigma) \leq idiam(\Delta_e) \leq k_r^i(d_X(\epsilon, g)) \\ &\leq k_r^i(\lceil d_X(\epsilon, \pi_{\Delta_w}(\tau)) \rceil + \zeta + 1) \leq k_r^i(\lceil d_{\Delta_w}(*, \tau)) \rceil + \zeta + 1) \text{ and} \\ \widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\sigma)) &= \widetilde{d}_X(\epsilon, \pi_{\Delta_e}(\sigma)) \leq ediam(\Delta_e) \leq k_r^e(d_X(\epsilon, g)) \\ &\leq k_r^e(\lceil \widetilde{d}_X(*, \pi_{\Delta_w}(\tau)) \rceil + \zeta + 1) \ . \end{aligned}$$

Therefore in all cases, we have $\widetilde{d}_{\Delta_w}(*, \sigma) \leq \mu^i(\widetilde{d}_{\Delta_w}(*, \tau))$ and $\widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\sigma)) \leq \mu^e(\widetilde{d}_X(\epsilon, \pi_{\Delta_w}(\tau)))$, as required. □

The tame filling inequality bounds in Theorem 4.2 are not sharp in general. In particular, we will improve upon these bounds for the example of almost convex groups in Section 5.5.

Recall from Section 1.3 that the group G is algorithmically stackable if there is a stacking $(\mathcal{N}, c)$ over a finite generating set A of G for which the subset

$S_c = \{(w, a, x) \mid w \in A^*, a \in A, x = c'(e_{w,a})\}$

of $A^* \times A \times A^*$ is computable, where $e_{w,a}$ denotes the directed edge in X labeled a from w to wa, and $c'(e_{w,a}) = c(e_{w,a})$ for $e_{w,a} \in \vec{E}_r$ and $c'(e_{w,a}) = a$ for $e_{w,a} \in \vec{E}_d$. For algorithmically stackable groups, the procedure described above for building a recursive combed normal filling from the stacking is again algorithmic.

Note that whenever the group G admits a tame filling inequality for a function $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$, and $g : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ satisfies the property that $f(n) \leq g(n)$ for all $n \in \mathbb{N}[\frac{1}{4}]$, then G also admits the same type of tame filling inequality for the function g. Applying this, we obtain a computable bound on tame filling inequalities for algorithmically stackable groups.

Theorem 4.3. *If G is an algorithmically stackable group, then G satisfies both intrinsic and extrinsic tame filling inequalities with respect to a recursive function.*

Proof. From Theorem 4.2, it suffices to show that the functions $k_{\mathcal{N}}^i$, $k_{\mathcal{N}}^e$, k_r^i, and k_r^e are bounded above by recursive functions.

We can write the functions

$$\begin{aligned} k_{\mathcal{N}}^i(n) &= \max\{l(y) \mid y \in \mathcal{N}_n\} \quad \text{and} \\ k_{\mathcal{N}}^e(n) &= \max\{d_X(\epsilon, x) \mid x \in P_n\} \end{aligned}$$

where $\mathcal{N}_n := \{y_g \in \mathcal{N} \mid g \in G \text{ and } d_X(\epsilon, g) \leq n\}$ and P_n is the set of prefixes of words in $\mathcal{N}_n$. Now for any prefix x of a word $y \in \mathcal{N}_n$, we have $l(x) \leq l(y)$, and so we can also write

$$k_{\mathcal{N}}^i(n) = \max\{l(x) \mid x \in P_n\}.$$

Since distance in a van Kampen diagram Δ always gives an upper bound for distance, via the map π_Δ, in the Cayley complex X, then for all $n \in \mathbb{N}$, we have $k_{\mathcal{N}}^e(n) \leq k_{\mathcal{N}}^i(n)$.

Moreover, $idiam(\Delta)$ must always be an upper bound for $ediam(\Delta)$, and so $k_r^e(n) \leq k_r^i(n)$ for all n. Thus it suffices to find recursive upper bounds for $k_{\mathcal{N}}^i$ and k_r^i.

For each word w over A, a stacking reduction algorithm for computing the the associated word y_w in $\mathcal{N}$ was given in Section 1.3. The set of words $\mathcal{N}_n$ is also the set $\mathcal{N}_n = \{y_u \mid u \in \cup_{i=0}^n A^i\}$ of normal forms for words of length up to n. By enumerating the finite set of words of length at most n, computing their normal forms in $\mathcal{N}$ with the reduction algorithm, and taking the maximum word length that occurs, we obtain $k_{\mathcal{N}}^i(n)$. Hence the function $k_{\mathcal{N}}^i$ is computable.

Given $w \in A^*$ and $a \in A$, we compute the two words y_w and y_{wa} and store them in a set L_e. Next we follow the definition of the normal form diagram Δ_e for the edge $e = e_{w,a}$ in the recursive construction of the normal filling from Section 1.3. If $(w,a,a) \in S_c$, then $e \in \vec{E}_d$ and we add no other words to L_e. On the other hand, if $(w,a,a) \notin S_c$, then $e \in \vec{E}_r$. In the latter case, by enumerating the finitely many words $x \in c(\vec{E}_r)$, and checking whether or not (w,a,x) lies in the computable set S_c, we can compute the word $c(e) = x$. Write $x = a_1 \cdots a_n$ with each $a_i \in A$. For $1 \leq i \leq n$, we compute the normal forms y_i in $\mathcal{N}$ for the words $wa_1 \cdots a_i$, and add these words to the set L_e. For each pair (y_{i-1}, a_i), we determine the word x_i such that $(y_i, a_i, x_i) \in S_c$. If $x_i \neq a_i$, we write $x_i = b_1 \cdots b_m$ with each $b_j \in A$, and add the normal forms for the words $y_{i-1}b_1 \cdots b_j$ to L_e for each j. Repeating this process through all of the steps in the construction of Δ_e, we must, after finitely many steps, have no more words to add to L_e. The set L_e now contains the normal form $y_{\pi_{\Delta_e}(v)}$ for each vertex v of the diagram Δ_e. Calculate $k(w,a) := \max\{l(y) \mid y \in L_e\}$.

Now as in Remark 1.10, for each vertex v of the normal form diagram Δ_e there is a path in Δ_e from the basepoint to v labeled by a word in the set L_e. Then $idiam(\Delta_e) \leq k(w,a)$, and we have an algorithm to compute $k(w,a)$.

Now we can write $k_r^i(n) \leq k_r'(n)$ for all $n \in \mathbb{N}$, where

$$k_r'(n) \quad := \quad \max\{k(w,a) \mid w \in \cup_{i=0}^n A^i, a \in A\}.$$

Repeating the computation of $k(w,a)$ above for all words w of length at most n and all $a \in A$, we can compute this upper bound k_r' for k_r^i, as required. $\square$

Remark 4.4. Although the proof of Theorem 4.3 shows in the abstract that an algorithm must exist to compute $k_r'(n)$, this proof does not give a method to find this algorithm starting from the computable set S_c. In particular, although every finite set is recursively enumerable, it is not clear how to enumerate the finite set $c(\vec{E}_r)$. In practice, however, for every example we will discuss, we start with both a finite presentation $\langle A \mid R \rangle$ for the group G and a stacking that (re)produces that presentation. In that case, the set $c(\vec{E}_r)$ must be contained in the finite set $R' := \{x \in A^* \mid \exists a \in A \text{ with } xa \in R\}$. Then we can replace the enumeration of $c(\vec{E}_r)$ with an enumeration of R', which can be computed from R.

5. Examples of stackable groups, and their tame filling inequalities

5.1. **Groups admitting complete rewriting systems.**

Recall that a *finite complete rewriting system* (finite *CRS*) for a group G consists of a finite set A and a finite set of rules $R \subseteq A^* \times A^*$ (with each $(u,v) \in R$ written $u \to v$) such that as a monoid, G is presented by $G = Mon\langle A \mid u = v$ whenever $u \to v \in R\rangle$, and the rewritings $xuy \to xvy$ for all $x, y \in A^*$ and $u \to v$ in R satisfy:

- *Normal forms:* Each $g \in G$ is represented by exactly one *irreducible* word (i.e. word that cannot be rewritten) over A.
- *Termination:* The (strict) partial ordering $x > y$ if $x \to x_1 \to ... \to x_n \to y$ is well-founded. ($\nexists$ infinite chain $w \to x_1 \to x_2 \to \cdots$.)

Given any finite CRS (A,R) for G, there is another finite CRS (A,R') for G with the same set of irreducible words such that the CRS is *minimal*. That is, for each $u \to v$ in R', the word v and all proper subwords of the word u are irreducible (see, for example, [19, p. 56]). Let A' be the closure of A under inversion. For each letter $a \in A' \setminus A$, there is an irreducible word $z_a \in A^*$ with $a =_G z_a$. Let $R'' := R' \cup \{a \to z_a \mid a \in A' \setminus A\}$. Then (A', R'') is also a minimal finite CRS for G, again with the same set of irreducible normal forms as the original CRS (A,R). For the remainder of this paper, we will assume that all of our complete rewriting systems are minimal and have an inverse-closed alphabet.

For any complete rewriting system (A,R), there is a natural associated symmetrized group presentation $\langle A \mid R'\rangle$, where R' is the closure of the relator set $\{uv^{-1} \mid u \to v \in R\} \cup \{aa^{-1} \mid a \in A'\}$ under free reduction (except the empty word), inversion, and cyclic conjugation.

Given any word $w \in A^*$, we write $w \xrightarrow{*} w'$ if there is any sequence of rewritings $w = w_0 \to w_1 \to \cdots \to w_n = w'$ (including the possibility that $n = 0$ and $w' = w$). A *prefix rewriting* of w with respect to the complete rewriting system (A,R) is a sequence of rewritings $w = w_0 \to \cdots \to w_n = w'$, written $w \xrightarrow{p*} w'$, such that at each w_i, the shortest possible reducible prefix is rewritten to obtain w_{i+1}. When w_n is irreducible, the number n is the *prefix rewriting length* of w, denoted $prl(w)$.

In [13] Hermiller and Meier constructed a diagrammatic 1-combing associated to a finite complete rewriting system. In Theorem 5.1, we utilize the analog of their construction to build a stacking from a finite CRS (whose canonical 1-combing built from the recursive combed normal filling is the one defined in [13]).

Theorem 5.1. *If the group G admits a finite complete rewriting system, then G is regularly stackable.*

Proof. Let $\mathcal{N} = \{y_g \mid g \in G\}$ be the set of irreducible words from a minimal finite CRS (A,R) for G, where A is inverse-closed. Then $\mathcal{N} = A^* \setminus \cup_{u \to v \in R} A^* u A^*$ is a regular language. Let Γ be the Cayley graph for the pair (G,A). Note that prefixes of irreducible words are also irreducible, and so $\mathcal{N}$ is a prefix-closed set of normal forms for G over A.

As usual, whenever e is a directed edge in Γ with label a and initial vertex g, then e lies in the set $\vec{E}_d$ of degenerate edges if and only if $y_g a y_{ga}^{-1}$ freely reduces to the empty word, and otherwise $e \in \vec{E}_r$.

Given a directed edge $e \in \vec{E}_r$ with initial vertex g and label a, the word $y_g a$ is reducible (since this edge is not in $\vec{E}_d$). Since y_g is irreducible, the shortest reducible prefix of $y_g a$ is the entire word. Minimality of the rewriting system R implies that there is a unique factorization $y_g = w\tilde{u}$ such that $\tilde{u}a$ is the left hand side of a unique rule $\tilde{u}a \to v$ in R; that is, $y_g a \to wv$ is a prefix rewriting. Define $c(e) := \tilde{u}^{-1}v$.

Property (S1) of the definition of stacking is immediate. To check property (S2), we first let p be the path in Γ that starts at g and follows the word $c(e)$. Since the word $\tilde{u}$ is a suffix x_g of the normal form y_g, then the edges in the path p that correspond to the letters in $\tilde{u}^{-1}$ all lie in the set $\vec{E}_d$ of degenerate edges. Hence we can choose $\tilde{c}_e = v$. For each directed edge e' in the subpath of p labeled by v, either e' also lies in $\vec{E}_d$, or else $e' \in \vec{E}_r$, and there is a factorization $v = v_1 a' v_2$ so that e' is the directed edge along p corresponding to the label $a' \in A$. In the latter case, if we denote the initial vertex of e' by g', then the prefix rewriting sequence from $y_{g'}a'v_2$ to its irreducible form is a (proper) subsequence of the prefix rewriting of $y_g a$. That is, if we define a function $prl : \vec{E}_r \to \mathbb{N}$ by $prl(e) := prl(y_g a)$ whenever e is an edge with initial vertex g and label a, we have $prl(e') < prl(e)$. Hence the ordering $<_c$ corresponding to our function $c : \vec{E}_r \to A^*$ satisfies the property that $e' <_c e$ implies $prl(e') < prl(e)$, and the well-ordering property on $\mathbb{N}$ implies that $<_c$ is a well-founded strict partial ordering. Thus (S2) holds as well.

The image set $c(\vec{E}_r)$ is the set of words $c(\vec{E}_r) = \{\tilde{u}^{-1}v \mid \exists a \in A \text{ with } \tilde{u}a \to v \text{ in } R\}$. Thus Property (S3) follows from finiteness of the set R of rules in the rewriting system. We now have a tuple $(\mathcal{N}, c)$ of data satisfying properties (S1-S3) of Definition 1.2, i.e., a stacking. The stacking presentation in this case is the symmetrized presentation associated to the rewriting system.

To determine whether a tuple (w, a, x) lies in the associated set S_c, we begin by computing the normal forms y_w and y_{wa} from w and wa, using the rewriting rules of our finite system. Then $(w, a, x) \in S_c$ if and only if either at least one of the words $y_w a$ and $y_{wa}a^{-1}$ is irreducible and $a = x$, or else both of the words $y_w a$ and $y_{wa}a^{-1}$ are reducible and there exist both a factorization $y_w = z\tilde{u}$ for some $z \in A^*$ and a rule $\tilde{u}a \to v$ in R such that $x = \tilde{u}^{-1}v$. Since there are only finite many rules in R to check for such a decomposition of y_w, it follows that the set S_c is also computable, and so this stacking is algorithmic. □

Theorem 4.3 now shows that any group with a finite complete rewriting system admits intrinsic and extrinsic tame filling inequalities with respect to a recursive function. By relaxing the bounds on tame filling inequalities further, we can write bounds on filling inequalities in terms of another important function in the study of rewriting systems.

Definition 5.2. *The* string growth complexity *function* $\gamma : \mathbb{N} \to \mathbb{N}$ *associated to a finite complete rewriting system* (A, R) *is defined by*

$$\gamma(n) := \max\{l(x) \mid \exists\ w \in A^* \text{ with } l(w) \le n \text{ and } w \xrightarrow{*} x\}$$

This function γ is an upper bound for the intrinsic (and hence also extrinsic) diameter function of the group G presented by the rewriting system. In the following, we show that G also satisfies tame filling inequalities with respect to a function Lipschitz equivalent to γ.

Corollary 5.3. *Let G be a group with a finite complete rewriting system. Let γ be the string growth complexity function for the associated minimal system and let ζ denote the length of the longest rewriting rule for this system. Then G satisfies both intrinsic and extrinsic tame filling inequalities for the recursive function $n \mapsto \gamma(\lceil n\rceil + \zeta + 2) + 1$.*

Proof. Let (A, R) be a minimal finite complete rewriting system for G such that A is inverse-closed. Let $(\mathcal{N}, c)$ be the stacking for G constructed in the proof of Theorem 5.1, and let X be the Cayley complex of the stacking presentation $\mathcal{P}$. Let $\mathcal{E} = \{\Delta_e, \Theta_e) \mid e \in E(X)\}$ be the associated recursive combed normal filling(where we note that the choice of subword $\tilde{c}_e$ of $c(e)$ for each $e \in \vec{E}_r$, used in the construction of Θ_e, is given in the proof of Theorem 5.1). For the rest of this proof, we rely heavily on the result and notation developed in the proof of Theorem 4.3 to obtain the tameness bounds for these edge homotopies.

From that proof, we have $k^e_{\mathcal{N}}(n) \le k^i_{\mathcal{N}}(n) = \max\{l(y) \mid y \in \mathcal{N}_n\}$ for all n, where $\mathcal{N}_n$ is the set of irreducible normal forms obtained by rewriting words over A of length at most n. Therefore $k^e_{\mathcal{N}}(n) \le k^i_{\mathcal{N}}(n) \le \gamma(n)$.

Also from that earlier proof, we have $k^e_r(n) \le k^i_r(n) \le k'_r(n)$ for all $n \in \mathbb{N}$. Suppose that $w \in A^*$ is a word of length at most n, $a \in A$, and $e = e_{w,a}$ is the directed edge in X from w to wa labeled by a. In this case we analyze the van Kampen diagram Δ_e more carefully. This diagram is built by successively applying prefix rewritings to the word $y_w a$ and/or by applying free reductions (which must also result from prefix rewritings). Hence for every vertex v in the diagram Δ_e, there is a path from the basepoint $*$ to v labeled by an irreducible prefix y of a word $x \in A^*$ such that $y_w a \overset{p*}{\to} x$, and this word y is the element of the set L_e corresponding to the vertex v. Then the maximum $k(w, a)$ of the lengths of the elements of L_e is bounded above by $\max\{l(y) \mid y \text{ is a prefix of } x \text{ and } y_w a \overset{p*}{\to} x\}$. Since the length of a prefix of a word x is at most $l(x)$, we have $k(w, a) \le \max\{l(x) \mid y_w a \overset{p*}{\to} x\}$.

Plugging this into the formula for k'_r, we obtain

$$\begin{aligned} k'_r(n) &= \max\{k(w,a) \mid w \in \cup_{i=0}^n A^i, a \in A, e_{w,a} \in \vec{E}_r\} \\ &\le \max\{l(x) \mid \exists\ w \in \cup_{i=0}^n A^i, a \in A, y_w a \overset{p*}{\to} x\}. \end{aligned}$$

Now for each word w of length at most n and each $a \in A$, we have $wa \overset{p*}{\to} y_w a$, and so

$k'_r(n) \le \max\{l(x) \mid \exists\ w \in \cup_{i=0}^n A^i, a \in A \text{ with } wa \overset{p*}{\to} x\} \le \gamma(n + 1)$.

Putting these inequalities together, we obtain $\mu^e(n) \le \mu^i(n)$ and

$$\begin{aligned} \mu^i(n) &= \max\{k^i_{\mathcal{N}}(\lceil n\rceil + 1) + 1, n + 1, k^i_r(\lceil n\rceil + \zeta + 1)\} \\ &\le \gamma(\lceil n\rceil + \zeta + 2) + 1. \end{aligned}$$

□

Remark 5.4. We note that every instance of rewriting in the proofs in this Section was a prefix rewriting, and so G also satisfies tame filling inequalities with γ replaced by the potentially smaller *prefix rewriting string growth complexity* function $\gamma_p(n) = \max\{l(x) \mid \exists\ w \in A^* \text{ with } l(w) \le n \text{ and } w \overset{p*}{\to} x\} \le \gamma(n)$.

5.2. **Thompson's group F.**

Thompson's group

$$F = \langle x_0, x_1 \mid [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2]\rangle$$

is the group of orientation-preserving piecewise linear homeomorphisms of the unit interval [0,1], satisfying that each linear piece has a slope of the form 2^i for some $i \in \mathbb{Z}$, and all breakpoints occur in the 2-adics. In [6], Cleary, Hermiller, Stein, and Taback show that Thompson's group with the generating set $A = \{x_0^{\pm 1}, x_1^{\pm 1}\}$ is stackable, with stacking presentation given by the symmetrization of the presentation above. Moreover, in [6, Definition 4.3] they give an algorithm for computing the stacking map, which can be shown to yield an algorithmic stacking for F.

Although we will not repeat their proof here, we describe the normal form set $\mathcal{N}$ associated to the stacking constructed for Thompson's group in [6] in order to discuss its formal language theoretic properties. Given a word w over the generating set $A = \{x_0^{\pm 1}, x_1^{\pm 1}\}$, denote the number of occurrences in w of the letter x_0 minus the number of occurrences in w of the letter x_0^{-1} by $expsum_{x_0}(w)$; that is, the exponent sum for x_0. The authors of that paper show ([6, Observation 3.6(1)]) that the set

$$\begin{aligned}\mathcal{N} := \{w \in A^* \mid \forall \eta \in \{\pm 1\}, &\text{ the words } x_0^{\eta}x_0^{-\eta},\ x_1^{\eta}x_1^{-\eta}, \text{ and } x_0^2x_1^{\eta} \text{ are not subwords of } w,\\ &\text{and } \forall \text{ prefixes } w' \text{ of } w, expsum_{x_0}(w') \leq 0\},\end{aligned}$$

is a set of normal forms for F. Moreover, each of these words labels a (6,0)-quasi-geodesic path in the Cayley complex X [6, Theorem 3.7].

This set $\mathcal{N}$ is the intersection of the regular language $A^* \setminus \cup_{u \in U} A^*uA^*$, where $U := \{x_0x_0^{-1}, x_0^{-1}x_0, x_1x_1^{-1}, x_1^{-1}x_1, x_0^2x_1, x_0^2x_1^{-1}\}$, with the language $L := \{w \in A^* \mid \forall$ prefixes w' of $w, expsum_{x_0}(w') \leq 0\}$. We refer the reader to the text of Hopcroft and Ullman [15] for definitions and results on context-free and regular languages we now use to analyze the set L. The language L can be recognized by a deterministic push-down automaton (PDA) which pushes an x_0^{-1} onto its stack whenever an x_0^{-1} is read, and pops an x_0^{-1} off of its stack whenever an x_0 is read. When $x_1^{\pm 1}$ is read, the PDA does nothing to the stack, and does not change its state. The PDA remains in its initial state unless an x_0 is read when the only symbol on the stack is the stack start symbol Z_0, in which case the PDA transitions to a fail state (at which it must then remain upon reading the remainder of the input word). Ultimately the PDA accepts a word whenever its final state is its initial state. Consequently, L is a deterministic context-free language. Since the intersection of a regular language with a deterministic context-free language is deterministic context-free, the set $\mathcal{N}$ is also a deterministic context-free language.

The authors of [6] construct the stacking of F as a stepping stone to showing that F with this presentation also admits a radial tame combing inequality with respect to a linear function. We note that although the definition of diagrammatic 1-combing is not included in that paper, and the coarse distance definition differs slightly, the constructions of 1-combings in the proofs are diagrammatic and admit Lipschitz equivalent radial tame combing inequality functions. Hence by Proposition 3.6, this group satisfies a linear extrinsic tame filling inequality.

Let $\mathcal{E}$ be the recursive combed normal filling associated to the stacking in [6], and let $\mathcal{D} = \{(\Delta_w, \Phi_w) \mid w \in A^*, w =_F \epsilon\}$ be the combed filling induced by $\mathcal{E}$ by the seashell procedure. As noted above, the van Kampen homotopies in the collection $\mathcal{D}$ are extrinsically f-tame for a linear function f. A consequence of Remark 1.10 and the seashell construction is that for each word $w \in A^*$ with $w =_F \epsilon$ and for each vertex v in Δ_w, there is a path in Δ_w from the basepoint $*$ to the vertex v labeled by the (6,0)-quasi-geodesic normal form in $\mathcal{N}$ representing $\pi_{\Delta_w}(v)$. Then we have $d_{\Delta_w}(*, v) \leq 6d_X(\epsilon, \pi_{\Delta_w}(v))$. Let $\widetilde{j} : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be the (linear) function defined by $\widetilde{j}(n) = 6\lceil n\rceil + 1$. Theorem 2.2 then shows that Thompson's group F also satisfies a linear intrinsic tame filling inequality, for the linear function $\widetilde{j} \circ f$.

On the other hand, we note that Cleary and Taback [7] have shown that Thompson's group F is not almost convex (in fact, Belk and Bux [1] have shown that F is not even minimally almost convex). Combining this with Theorem 5.6 below, Thompson's group F cannot satisfy an intrinsic or extrinsic tame filling inequality for the identity function.

5.3. **Iterated Baumslag-Solitar groups.**

The iterated Baumslag-Solitar group

$$G_k = \langle a_0, a_1, ..., a_k \mid a_i^{a_{i+1}} = a_i^2; 0 \leq i \leq k-1\rangle$$

admits a finite complete rewriting system for each $k \geq 1$ (first described by Gersten; see [12] for details), and so Theorem 5.1 shows that this group is regularly stackable.

Gersten [10, Section 6] showed that G_k has an isoperimetric function that grows at least as fast as a tower of exponentials

$$E_k(n) := \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{\text{k times}} .$$

It follows from his proof that the (minimal) extrinsic diameter function for this group is at least $O(E_{k-1}(n))$. Hence this is also a lower bound for the (minimal) intrinsic diameter function for this group. Then by Proposition 2.1, G_k cannot satisfy an intrinsic or extrinsic tame filling inequality for the function E_{k-2}. (In the extrinsic case, this was shown in the context of tame combings in [12].) Combining this with Corollary 5.3, for $k \geq 2$ the group G_k is an example of a regularly stackable group which admits intrinsic and extrinsic recursive tame filling inequalities but which cannot satisfy a tame filling inequality for E_{k-2}.

5.4. **Solvable Baumslag-Solitar groups.**

The solvable Baumslag-Solitar groups are presented by $G = BS(1,p) = \langle a, t \mid tat^{-1} = a^p\rangle$ with $p \in \mathbb{Z}$. In [6] Cleary, Hermiller, Stein, and Taback show that for $p \geq 3$, the groups $BS(1,p)$ admit a linear radial tame combing inequality, and hence (from Proposition 3.6) a linear extrinsic tame filling inequality.

We note that the combed filling in their proof is induced by the recursive combed normal filling associated to a regular stacking, which we describe here in order to obtain an intrinsic tame filling inequality for these groups. The set of normal forms over the generating set

$A = \{a, a^{-1}, t, t^{-1}\}$ is

$$\mathcal{N} := \{t^{-i}a^m t^k \mid i, k \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z}, \text{ and either } p \not| m \text{ or } 0 \in \{i, k\}\}.$$

The recursive edges in $\vec{E}_r = \vec{E}(X) \setminus \vec{E}_d$ are the directed edges of the form $e_{w,b}$ with initial point w and label $b \in A$ satisfying one of the following:

(1) $w = t^{-i}a^m$ and $b = t^\eta$ with $m \neq 0$, $\eta \in \{\pm 1\}$, and $-i + \eta \leq 0$, or
(2) $w = t^{-i}a^m t^k$ and $b = a^\eta$ with $k > 0$ and $\eta \in \{\pm 1\}$.

In case (1), we define $c(e_{t^{-i}a^m,t^\eta}) := (a^{-\nu p}ta^\nu)^\eta$, where $\nu := \frac{m}{|m|}$ is 1 if $m > 0$ and -1 if $m < 0$. In case (2) we define $c(e_{t^{-i}a^m t^k,a^\eta}) := t^{-1}a^{\eta p}t$.

Properties (S1) and (S3) of the definition of stacking follow directly. To show that the pair stacking map c also satisfies property (S2), we first briefly describe the Cayley complex X for the finite presentation above; see for example [8, Section 7.4] for more details. The Cayley complex X is homeomorphic to the product $\mathbb{R} \times T$ of the real line with a regular tree T, and there are canonical projections $\Pi_\mathbb{R} : X \to \mathbb{R}$ and $\Pi_T : X \to T$. The projection Π_T takes each edge labeled by an $a^{\pm 1}$ to a vertex of T. Each edge of T is the image of infinitely many t edges of X^1, with consistent orientation, and so we may consider the edges of T to be oriented and labeled by t, as well. For the normal form $y_g = t^{-i}a^m t^k \in \mathcal{N}$ of an element $g \in G$, the projection onto T of the path in X^1 starting at ϵ and labeled by y_g is the unique geodesic path, labeled by $t^{-i}t^k$, in the tree T from $\Pi_T(\epsilon)$ to $\Pi_T(g)$. For any directed edge e in $\vec{E}_r$ in case (2) above, there are $p + 1$ 2-cells in the Cayley complex X that contain e in their boundary, and the path $c(e)$ starting from the initial vertex of e is the portion of the boundary, disjoint from e, of the only one of those 2-cells σ that satisfies $d_T(\Pi_T(\epsilon), \Pi_T(q)) \leq d_T(\Pi_T(\epsilon), \Pi_T(e))$ for all points $q \in \sigma$, where d_T is the path metric in T. For any edge e' that lies both in this $c(e)$ path and in $\vec{E}_r$, then e' is again a recursive edge of type (2), and we have $d_T(\Pi_T(\epsilon), \Pi_T(e')) < d_T(\Pi_T(\epsilon), \Pi_T(e))$. Thus the well-ordering on $\mathbb{N}$ applies, to show that there are at most finitely many $e'' \in \vec{E}_r$ with $e'' <_c e$ in case (2).

The other projection map $\Pi_\mathbb{R}$ takes each vertex $t^{-i}a^m t^k$ to the real number $p^{-i}m$, and so takes each edge labeled by $t^{\pm 1}$ to a single real number, and takes each edge labeled $a^{\pm 1}$ to an interval in $\mathbb{R}$. For an edge $e \in \vec{E}_r$ in case (1) above, there are exactly two 2-cells in X containing e, and the path $c(e)$ starting at the initial vertex $w = t^{-i}a^m$ of e travels around the boundary of the one of these two cells (except for the edge e) whose image, under the projection $\Pi_\mathbb{R}$, is closest to 0. The only possibly recursive edge e' in this $c(e)$ path must also have type (1), and moreover the initial vertex of e' is $w' = t^{-i}a^{m-\nu}$ and satisfies $|\Pi_\mathbb{R}(w')| = |\Pi_\mathbb{R}(w)| - p^{-i}$. Then in case (1) also there are only finitely many recursive edges that are $<_c e$, completing the proof of property (S2).

Therefore the tuple $(\mathcal{N}, c)$ is a stacking, and the symmetrization of the presentation above is the stacking presentation. The canonical diagrammatic 1-combing built from the associated recursive combed normal filling is the 1-combing constructed in [6].

Let $\mathcal{D} = \{(\Delta_w, \Phi_w) \mid w \in A^*, w =_F \epsilon\}$ be the combed filling induced by this recursive combed normal filling via the seashell procedure. From Remark 1.10, we know that for each vertex v of a van Kampen diagram Δ_w in this collection, there is a path in Δ_w from $*$ to v labeled by the normal form of the element $\pi_{\Delta_w}(v)$ of $BS(1,p)$. The normal form

y_g of $g \in G$ can be obtained from a geodesic representative by applying (the infinite set of) rewriting rules of the form $ta^\eta \to a^{\eta p}t$ and $a^\eta t^{-1} \to t^{-1}a^{\eta p}$ for $\eta = \pm 1$ together with $t^{-1}a^{pm}t \to a^m$ for $m \in \mathbb{Z}$ and free reductions. Then $d_{\Delta_w}(*, v) \leq l(y_g) \leq j(d_X(\epsilon, \pi_{\Delta_w}(v))$ for the function $j : \mathbb{N} \to \mathbb{N}$ given by $j(n) = p^n$. Theorem 2.2 and the linear extrinsic tame filling inequality result above now apply, to show that the group $BS(1, p)$ with $p \geq 3$ also satisfies an intrinsic tame filling inequality with respect to a function $\mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ that is Lipschitz equivalent to the exponential function $n \mapsto p^n$ with base p.

5.5. **Almost convex groups.**

One of the original motivations for the definition of a radial tame combing inequality in [12] was to capture Cannon's [5] notion of almost convexity in a quasi-isometry invariant property. Let G be a group with an inverse-closed generating set A, and let d_Γ be the path metric on the associated Cayley graph Γ. For $n \in \mathbb{N}$, define the sphere $S(n)$ of radius n to be the set of points in Γ a distance exactly n from the vertex labeled by the identity ϵ. Recall that the ball $B(n)$ of radius n is the set of points in Γ whose path metric distance to ϵ is less than or equal to n.

Definition 5.5. *A group G is* almost convex *with respect to the finite symmetric generating set A if there is a constant k such that for all $n \in \mathbb{N}$ and for all g, h in the sphere $S(n)$ satisfying $d_\Gamma(g, h) \leq 2$ (in the corresponding Cayley graph), there is a path inside the ball $B(n)$ from g to h of length no more than k.*

Cannon [5] showed that every group satisfying an almost convexity condition over a finite generating set is also finitely presented. Thiel [20] showed that almost convexity is a property that depends upon the finite generating set used.

In Theorem 5.6, we show that a pair (G, A) that is almost convex is algorithmically stackable and (applying Theorem 7.1) must also lie in the quasi-isometry invariant class of groups admitting linear intrinsic and extrinsic tame filling inequalities. Moreover almost convexity of (G, A) is exactly characterized by admitting a finite set R of defining relations for G over A such that the pair $(G, \langle A \mid R \rangle)$ satisfies an intrinsic or extrinsic tame filling inequality with respect to the identity function $\iota : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ (i.e. $\iota(n) = n$ for all n). In the extrinsic case, equivalence of almost convexity and an extrinsic tame normal inequality for ι follows directly from the equivalence of almost convexity with a radial tame combing inequality for the identity shown by Hermiller and Meier in [12, Theorem C], together with Proposition 3.6. We give some details here which include a description of the stacking involved, and a minor correction to the proof in that earlier paper.

Theorem 5.6. *Let G be a group with finite generating set A, and let $\iota : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ denote the identity function. The following are equivalent:*

(1) *The pair (G, A) is almost convex*
(2) *There is a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ for G that satisfies an intrinsic tame filling inequality with respect to ι.*
(3) *There is a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ for G that satisfies an extrinsic tame filling inequality with respect to ι.*

Moreover, if any of these hold, then G is algorithmically stackable over A.

Proof. Suppose that the group G has finite symmetric generating set A, and let Γ be the corresponding Cayley graph.

Almost convex $\Rightarrow$ algorithmically stackable:

Suppose that the group G is almost convex with respect to A, with an almost convexity constant k. Let $\mathcal{N} = \{z_g \mid g \in G\}$ be the set of shortlex normal forms over A for G. Let $\vec{E}_d$ be the corresponding set of directed degenerate edges, and let $\vec{E}_r = \vec{E}(\Gamma) \setminus \vec{E}_d$ be the set of recursive edges.

Let e be any element of $\vec{E}_r$ and suppose that e is oriented from endpoint g to endpoint h. If $\widetilde{d_\Gamma}(\epsilon, g) = \widetilde{d_\Gamma}(\epsilon, h) = n$, then the points g and h lie in the same sphere. Almost convexity of (G, A) implies that there is a directed edge path in X from g to h of length at most k that lies in the ball $B(n)$. We define $\tilde{c}_e = c(e)$ to be the shortlex least word over A that labels a path in $B(n)$ from g to h. If $\widetilde{d_\Gamma}(\epsilon, g) = n$ and $\widetilde{d_\Gamma}(\epsilon, h) = n + 1$, then we can write $z_h =_{A^*} z_{h'}b$ for some $h' \in G$ and $b \in A$. Hence $g, h' \in S(n)$ and $d_\Gamma(g, h') \leq 2$. Again in this case we define $\tilde{c}_e$ to be the shortlex least word over A that labels a path in X of length at most k inside of the ball $B(n)$ from g to h'. The almost convexity property shows that the word $c(e) := \tilde{c}(e)b$ has length at most $k + 1$, this word labels a path from g to h, and $c(e)$ decomposes as the word $\tilde{c}(e)$ followed by a suffix $x_h = b$ of z_h. Similarly, if $\widetilde{d_\Gamma}(\epsilon, g) = n + 1$ and $\widetilde{d_\Gamma}(\epsilon, h) = n$, then $z_g = z_{g'}b$ for some $b \in A$ and $g' \in G$, and we define $\tilde{c}_e$ to be the shortlex least word labeling a path in $B(n)$ from g' to h. Then $c(e) := b^{-1}\tilde{c}_e$ labels a path from g to h, and decomposes as a prefix b^{-1}, that is the inverse of a suffix $x_g = b$ of z_g, followed by $\tilde{c}_e$.

In each of these three cases, for any point p in the interior of e, we have $\widetilde{d_\Gamma}(\epsilon, p) = n + \frac{1}{2}$. For any directed edge e' that lies both in $\vec{E}_r$ and in the path of Γ starting at g and labeled by $c(e)$, the edge e' must lie in the subpath labeled by $\tilde{c}(e)$, and hence e' is contained in $B(n)$. Therefore any point p' in the interior of e' must satisfy $\widetilde{d_X}(\epsilon, p') \leq n - \frac{1}{2} < \widetilde{d_X}(\epsilon, p)$. That is, if we define the function $f_\Gamma : \vec{E}(\Gamma) \to \mathbb{N}[\frac{1}{4}]$ by $f_\Gamma(u) := \widetilde{d_\Gamma}(\epsilon, q)$ for any (and hence all) $q \in Int(u)$, we have that $e' <_c e$ implies $f_\Gamma(e') < f_\Gamma(e)$ in the standard well-ordering on $\mathbb{N}[\frac{1}{4}]$. Hence the relation $<_c$ is a well-founded strict partial ordering. Properties (S1) and (S2) of Definition 1.2 hold for the function c.

The image set $c(\vec{E}_r)$ of this function c is contained in the finite set of all nonempty words over A of length up to $k+1$ that represent the identity element of G, and so (S3) also holds. We now have that the tuple $(\mathcal{N}, c)$ is a stacking.

We are left with showing computability for the set S_c defined by $S_c = \{(w, a, x) \mid c'(e_{w,a}) = x\} \subset A^* \times A \times A^*$ where $e_{w,a}$ denotes the edge in Γ from w to wa, and $c'(e_{w,a}) = c(e_{w,a})$ for all $e_{w,a} \in \vec{E}_r$, and $c'(e_{w,a}) = a$ for all $e_{w,a} \in \vec{E}_d$. Suppose that (w, a, x) is any element of $A^* \times A \times A^*$. Cannon [5, Theorem 1.4] has shown that the word problem is solvable for G, and so by enumerating the words in A^* in increasing shortlex order, and checking whether each in turn is equal in G to w, we can find the shortlex normal

form z_w for w. Similarly we compute z_{wa}. If the word $z_w a z_{wa}^{-1}$ freely reduces to 1, then the tuple (w, a, x) lies in S_c if and only if $x = a$.

Suppose on the other hand that the word $z_w a z_{wa}^{-1}$ does not freely reduce to 1. If $l(z_w) = l(z_{wa})$ is the natural number n, then we enumerate the elements of the finite set $\cup_{i=0}^{n} A^i$ of words of length up to n in increasing shortlex order. For each word $y = a_1 \cdots a_m$ in this enumeration, with each $a_i \in A$, we use the word problem solution again to compute the word length $l_{y,i}$ of the normal form $z_{wa_1 \cdots a_i}$ for each $0 \le i \le m$. If each $l_{y,i} \le n$, and equalities $l_{y,i} = n$ do not hold for two consecutive indices i, then (w, a, y) lies in S_c and we halt the enumeration; otherwise, we go on to check the next word in our enumeration. The tuple (w, a, x) lies in S_c if and only if x is the unique word y that results when this algorithm stops. The cases that $l(z_w) = l(z_{wa}) \pm 1$ are similar.

Combining the algorithms in the previous two paragraphs, we have that the set S_c is computable.

(1) implies (3):

Suppose that the group G is almost convex with respect to A, with almost convexity constant k. Let $(\mathcal{N}, c)$ be the stacking obtained above, and let X be the Cayley complex for the stacking presentation $\mathcal{P} = \langle A \mid R_c \rangle$, with 1-skeleton $X^1 = \Gamma$. Let $\mathcal{E} = \{(\Delta_e, \Theta_e) \mid e \in E(X)\}$ be the set of normal form diagrams and edge homotopies from the associated recursive combed normal filling.

Theorem 4.2 can now be applied, but unfortunately this result is insufficient. Although the fact that all of the normal forms in $\mathcal{N}$ are geodesic implies that the functions $k_{\mathcal{N}}^i$ and k_r^i are the identity, the tame filling inequality bounds μ^i and μ^e are not. Instead, we follow the steps of the algorithm that built the recursive combed normal filling more carefully.

Let e be any edge of X, again with endpoints g and h, and let $n := \min\{d_X(\epsilon, g), d_X(\epsilon, h)\}$; that is, either $g, h \in S(n)$, or one of these points lies in $S(n)$ and the other is in $S(n+1)$. Let $\hat{e}$ be the edge corresponding to e in the van Kampen diagram Δ_e, and let p be an arbitrary point in $\hat{e}$.

Case I. Suppose that $e \in E_d$. Then Δ_e is a line segment with no 2-cells, and the path $\pi_{\Delta_e} \circ \Theta_e(p, \cdot)$ follows a geodesic in X^1. Hence this path is extrinsically ι-tame.

Case II. Suppose that $e \in E(X) \setminus E_d$. We prove this case by Noetherian induction. By construction, the paths $\pi_{\Delta_e} \circ \Theta_e(\hat{g}, \cdot)$ and $\pi_{\Delta_e} \circ \Theta_e(\hat{h}, \cdot)$ follow the geodesic paths in X starting from ϵ and labeled by the words y_g and y_h at constant speed.

Suppose that p is a point in the interior of $\hat{e}$. We follow the notation of the recursive construction of Θ_e in Section 4. In that construction, edge homotopies are constructed for directed edges; by slight abuse of notation, let e also denote the directed edge from g to h that yields the element (Δ_e, Θ_e) of $\mathcal{E}$. Recall that this recursive procedure utilizes a factorization of the word $c(e)$ as $c(e) = x_g \tilde{c}_e x_h$. In our definition of $c(e)$ above, we defined this factorization so that for each edge e' (no matter whether e' is in $\vec{E}_d$ or $\vec{E}_r$) in the $\tilde{c}_e$ path, we have $f_\Gamma(e') < f_\Gamma(e)$. On the interval $[0, a_p]$, the path $\Theta_e(p, \cdot)$ follows a path $\Theta_i(\Xi_e(p, 0), \cdot)$ in a subdiagram of Δ_e that is either an edge homotopy for an edge e_i of X that lies in this $\tilde{c}_e$ subpath, or a line segment labeled by a shortlex normal form. Hence either by induction or case I, the homotopy Θ_i is extrinsically ι-tame.

On the interval $[a_p, 1]$, the path $\Theta_e(p, \cdot)$ follows the path $\Xi_e(p, \cdot)$ from the point $\Xi_e(p, 0)$ (in the subpath of ∂f_e labeled $\tilde{c}_e$, whose image in X is contained in $B(n)$) through the interior of the 2-cell f_e of Δ_e to the point p. We have $\widetilde{d}_X(\epsilon, \pi_{\Delta_e}(\Xi_e(p,0))) \leq n$, $\widetilde{d}_X(\epsilon, \pi_{\Delta_e}(\Xi_e(p,t))) = n + \frac{1}{4}$ for all $t \in (0,1)$, and $\widetilde{d}_X(\epsilon, \pi_{\Delta_e}(\Xi_e(p,1))) = \widetilde{d}_X(\epsilon, p) = f_\Gamma(e) = n + \frac{1}{2}$. Hence the path $\Xi_e(p, \cdot)$ is extrinsically ι-tame. Putting these pieces together, we have that Θ_e is also extrinsically ι-tame in Case II.

Thus in the recursive combed normal filling $(\mathcal{N}, \mathcal{E})$, each edge homotopy is extrinsically ι-tame, and hence the same is true for the van Kampen homotopies of the recursive combed filling $(\mathcal{N}, \mathcal{D})$ induced by $\mathcal{E}$, by the proof of (4) $\Rightarrow$ (1) in Proposition 3.6. Therefore $(G, \mathcal{P})$ satisfies an extrinsic tame filling inequality with respect to the same function ι.

(1) implies (2): As noted in Remark 1.10, the recursive combed filling constructed above from the almost convexity condition satisfies the property that for every vertex v in a van Kampen diagram Δ of $\mathcal{D}$, there is a path in Δ from $*$ to v labeled by the shortlex normal form for the element $\pi_\Delta(v)$ of G. Since these normal forms label geodesics in X, it follows that intrinsic and extrinsic distances (to the basepoints) in the diagrams Δ of $\mathcal{D}$ are the same. Thus the pair $(G, \mathcal{P})$ satisfies an intrinsic tame filling inequality with respect to the same function ι.

(2) or (3) implies (1): The proof of this direction in the extrinsic case closely follows the proof of [12, Theorem C], and the proof in the intrinsic case is quite similar. $\square$

Remark 5.7. As in Remark 4.4, Cannon's word problem algorithm for almost convex groups, which we applied in the proof of Theorem 5.6, requires the use of an enumeration of a finite set of words over A, namely those that represent ϵ in G and have length at most $k + 2$. As Cannon also points out [5, p. 199], although this set is indeed recursive, there may not be an algorithm to find this set, starting from (G, A) and the constant k.

Since every word hyperbolic group is almost convex, and the set of shortlex normal forms (used in the proof of Theorem 5.6 to construct a stacking for any word hyperbolic group) is a regular language, we have shown that every word hyperbolic group is regularly stackable. Combining Theorem 5.1 with a result of Hermiller and Shapiro [14], that the fundamental group of every closed 3-manifold with a uniform geometry other than hyperbolic must have a finite complete rewriting system, shows that these groups are regularly stackable as well. Hence we obtain the following.

Corollary 5.8. *If G is the fundamental group of a closed 3-manifold with a uniform geometry, then G is regularly stackable.*

6. Groups with a fellow traveler property and their tame filling inequalities

In this Section, we consider a class of finitely presented groups which admit a rather different procedure for constructing van Kampen diagrams. Let G be a group with a finite inverse-closed generating set A such that no element of A represents the identity ϵ of G, and let Γ be the Cayley graph of G over A. We also assume that G admits a set $\mathcal{N} = \{y_g \mid g \in G\}$ of simple word normal forms over A for G that satisfies a (synchronous)

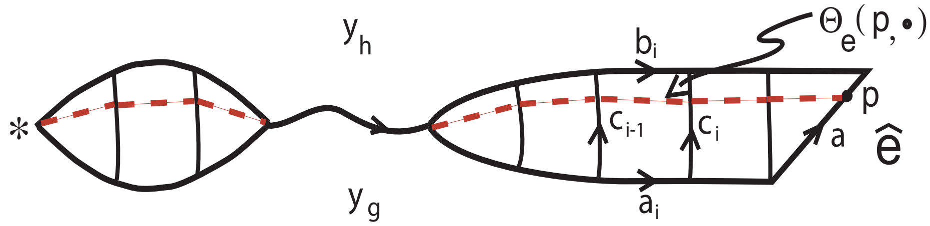

FIGURE 9. "Thin" van Kampen diagram Δ_e

K-fellow traveler property. That is, there is a constant $K \geq 1$ such that whenever $g, h \in G$ and $a \in A$ with $ga =_G h$, and we write $y_g = a_1 \cdots a_m$ and $y_h = b_1 \cdots b_n$ with each $a_i, b_i \in A$ (where, without loss of generality, we assume $m \leq n$), then for all $1 \leq i \leq m$ we have $d_\Gamma(a_1 \cdots a_i, b_1 \cdots b_i) \leq K$, and for all $m < i \leq n$ we have $d_X(g, b_1 \cdots b_i) \leq K$.

For each $m < i \leq n$, let a_i denote the empty word. Let c_0 denote the empty word, let $c_n := a$, and for each $1 \leq i \leq n-1$, let c_i be a word in A^* labeling a geodesic path in Γ from $a_1 \cdots a_i$ to $b_1 \cdots b_i$. Thus each c_i has length at most K.

This fellow traveler property implies that the set R of nonempty words over A of length up to $2K+2$ that represent the trivial element is a set of defining relators for G. Let $\mathcal{P} = \langle A \mid R \rangle$ be the symmetrized presentation for G, and let X be the Cayley complex.

A van Kampen diagram Δ_e for the word $y_g a y_h^{-1}$ corresponding to the edge e labeled a from g to h in X is built by successively gluing 2-cells labeled $c_{i-1} a_i c_i^{-1} b_{i-1}$, for $1 \leq i \leq n$, along their common c_i boundaries. Then the diagram Δ_e is "thin", in that it has only the width of (at most) one 2-cell. An edge homotopy Θ_e for this diagram can be constructed to go successively through each 2-cell in turn from the basepoint $*$ to the edge $\hat{e}$ corresponding to e; see Figure 9 for an illustration. Let $\mathcal{E} = \{(\Delta_e, \Theta_e)\}$ be the collection of these normal form diagrams and edge homotopies; the pair $(\mathcal{N}, \mathcal{E})$ is a combed normal filling.

Proposition 6.1. *Let G be a group with a finite generating set A and Cayley graph Γ. If G has a set $\mathcal{N}$ of simple word normal forms with a K-fellow traveler property such that the set*

$$S_n := \{w \in A^* \mid d_\Gamma(\epsilon, w) \leq n \text{ and } w \text{ is a prefix of a word in } \mathcal{N}\}$$

is a finite set for all $n \in \mathbb{N}$, then G satisfies both intrinsic and extrinsic tame filling inequalities for finite-valued functions.

Proof. We utilize the finite presentation $\mathcal{P}$ for G, with Cayley complex X, and the combed normal filling $(\mathcal{N}, \mathcal{E})$ constructed above. Let $\mathcal{D} = \{(\Delta_w, \Psi_w) \mid w \in A^*, w =_G \epsilon\}$ be the combed filling obtained from $\mathcal{E}$ using the seashell procedure. Also let Δ_w be any of the diagrams in $\mathcal{D}$, let p be any point in $\partial\Delta_w$, and let $0 \leq s < t \leq 1$.

Let $\hat{e}$ be an edge of $\partial\Delta_w$ containing p (where p may be in the interior or an endpoint). Then the path $\Psi_w(p, \cdot)$ lies in a subdiagram Δ_e of Δ_w such that Δ_e is the diagram in $\mathcal{E}$ corresponding to the edge $e = \pi_{\Delta_w}(\hat{e})$ of X, and $\Psi_w(p, \cdot) = \Theta_e(p, \cdot)$. Let $\hat{g}$ be an endpoint of $\hat{e}$, with $g = \pi_{\Delta_e}(\hat{g}) \in G$ an endpoint of e. Let y_g be the normal form of g in $\mathcal{N}$.

Applying the "thinness" of Δ_e, there is a path labeled y_g in $\partial\Delta_e$ starting at the basepoint, and every point of Δ_e lies in some closed cell of Δ_e that also contains a vertex in this boundary path. In particular, there are vertices v_s and v_t on the boundary path y_g of Δ_e such that the point $\Psi_w(p,s) = \Theta_e(p,s)$ and the point v_s occupy the same closed 0, 1, or 2-cell in Δ_e (and hence also in Δ_w), $\Psi_w(p,t) = \Theta_e(p,t)$ and v_t occupy a common closed cell, and v_s occurs before (i.e., closer to the basepoint) or at v_t along the y_g path. As usual let $\zeta \leq 2K+2$ denote the length of the longest relator in the presentation $\mathcal{P}$. Then we have $|\widetilde{d}_{\Delta_w}(*,\Psi_w(p,s)) - \widetilde{d}_{\Delta_w}(*,v_s)| \leq \zeta+1$ and $|\widetilde{d}_X(\epsilon,\pi_{\Delta_w}(\Psi_w(p,s))) - \widetilde{d}_X(\epsilon,\pi_{\Delta_w}(v_s))| \leq \zeta+1$, and similarly for the pair $\Psi_w(p,t)$ and v_t. Write the word $y_g = y_1y_2y_3$ where the vertex v_s occurs on the y_g path in $\partial\Delta_e \subset \Delta_w$ between the y_1 and y_2 subwords, and the vertex v_t between the y_2 and y_3 subwords. Note that y_1y_2 is a prefix of a normal form word in $\mathcal{N}$, and so satisfies $y_1y_2 \in S_{d_X(\epsilon,\pi_{\Delta_w}(v_t))}$.

Define the function $t^i : \mathbb{N} \to \mathbb{N}$ by $t^i(n) := \max\{l(w) \mid w \in S_n\}$. Since each $|S_n|$ is finite, this function is finite-valued. Utilizing the fact that t^i is a nondecreasing function, we have

$$\begin{aligned}
\widetilde{d}_{\Delta_w}(*,\Psi_w(p,s)) &\leq \widetilde{d}_{\Delta_w}(*,v_s)+\zeta+1 \quad\leq\quad l(y_1)+\zeta+1 \\
&\leq l(y_1y_2)+\zeta+1 \quad\leq\quad t^i(d_X(\epsilon,\pi_{\Delta_w}(v_t)))+\zeta+1 \\
&\leq t^i(\widetilde{d}_{\Delta_w}(*,v_t))+\zeta+1 \\
&\leq t^i(\lceil \widetilde{d}_{\Delta_w}(*,\Psi_w(p,t))\rceil+\zeta+1)+\zeta+1
\end{aligned}$$

Then G satisfies an intrinsic tame filling inequality for the function $n \to t^i(\lceil n\rceil+2K+3)+2K+3$.

Next define the function $t^e : \mathbb{N} \to \mathbb{N}$ by

$$t^e(n) := \max\{d_X(\epsilon,v) \mid v \text{ is a prefix of a word in } S_n\}.$$

Again, this is a finite-valued nondecreasing function. In this case, we note that since y_1 is a prefix of y_1y_2, then y_1 is a prefix of a word in $S_{d_X(\epsilon,\pi_{\Delta_w}(v_t))}$. Then

$$\begin{aligned}
\widetilde{d}_X(\epsilon,\pi_{\Delta_w}(\Psi_w(p,s))) &\leq \widetilde{d}_X(\epsilon,\pi_{\Delta_w}(v_s))+\zeta+1 \quad=\quad d_X(\epsilon,y_1)+\zeta+1 \\
&\leq t^e(d_X(\epsilon,\pi_{\Delta_w}(v_t)))+\zeta+1 \\
&\leq t^e(\lceil \widetilde{d}_X(\epsilon,\pi_{\Delta_w}(\Psi_w(p,t)))\rceil+\zeta+1)+\zeta+1.
\end{aligned}$$

Then G satisfies an extrinsic tame filling inequality for the function $n \to t^e(\lceil n\rceil+2K+3)+2K+3$. □

We highlight two special cases in which the hypothesis of Proposition 6.1, that each set S_n is finite, is satisfied. The first is the case in which the set of normal forms is prefix-closed. For this case, the functions $t^i = k^i_{\mathcal{N}}$ and $t^e = k^e_{\mathcal{N}}$ are the functions defined in Section 4, and so we have the following.

Corollary 6.2. *If G has a prefix-closed set of normal forms that satisfies a K-fellow traveler property, then G admits an intrinsic tame filling inequality for the function $f^i(n) = k^i_{\mathcal{N}}(\lceil n\rceil+2K+3)+2K+3$ and an extrinsic tame filling inequality for the function $f^e(n) = k^e_{\mathcal{N}}(\lceil n\rceil+2K+3)+2K+3$.*

The second is the case in which the set of normal forms is quasi-geodesic; that is, there are constants $\lambda, \lambda' \geq 1$ such that every word in this set is a (λ, λ')-quasi-geodesic. For a group G with generators A and Cayley graph Γ, a word $y \in A^*$ is a (λ, λ')-*quasi-geodesic* if whenever $y = y_1 y_2 y_3$, then $l(y_2) \leq \lambda d_\Gamma(\epsilon, y_2) + \lambda'$. Actually, we will only need a slightly weaker property, that this inequality holds whenever y_2 is a prefix of y (i.e., when $y_1 = 1$). In this case, the set S_n is a subset of the finite set $\cup_{i=0}^{\lambda n + \lambda'} A^i$ of words of length at most $\lambda n + \lambda'$. Then $t^e(n) \leq t^i(n) \leq \lambda n + \lambda'$ for all n. Putting these results together yields the following.

Corollary 6.3. *If a finitely generated group G admits a quasi-geodesic language of simple word normal forms satisfying a K-fellow traveler property, then G satisfies linear intrinsic and extrinsic tame filling inequalities.*

7. Quasi-isometry invariance for tame filling inequalities

In this section we show that, as with the diameter inequalities [4], [9], tame filling inequalities are also quasi-isometry invariants, up to Lipschitz equivalence of functions (and in the intrinsic case, up to sufficiently large set of defining relations). In the extrinsic case, this follows from Proposition 3.6 and the proof of Theorem [12, Theorem A], but with a slightly different definition of coarse distance. We include the details for both here, to illustrate the difference between the intrinsic and extrinsic cases.

Theorem 7.1. *Suppose that $(G, \mathcal{P})$ and $(H, \mathcal{P}')$ are quasi-isometric groups with finite presentations. If $(G, \mathcal{P})$ satisfies an extrinsic tame filling inequality with respect to f, then $(H, \mathcal{P}')$ satisfies an extrinsic tame filling inequality with respect to a function that is Lipschitz equivalent to f. If $(G, \mathcal{P})$ satisfies an intrinsic tame filling inequality with respect to f, then after adding all relators of length up to a sufficiently large constant to the presentation $\mathcal{P}'$, the pair $(H, \mathcal{P}')$ satisfies an intrinsic tame filling inequality with respect to a function that is Lipschitz equivalent to f.*

Proof. Write the finite presentations $\mathcal{P} = \langle A \mid R \rangle$ and $\mathcal{P}' = \langle B \mid S \rangle$; as usual we assume that these presentations are symmetrized. Let X be the 2-dimensional Cayley complex for the pair $(G, \mathcal{P})$, and let Y be the Cayley complex associated to $(H, \mathcal{P}')$. Let d_X, d_Y be the path metrics in X and Y (and hence also the word metrics in G and H with respect to the generating sets A and B), respectively.

Quasi-isometry of these groups means that there are functions $\phi : G \to H$ and $\theta : H \to G$ and a constant $k > 1$ such that for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$, we have

(1) $\frac{1}{k} d_X(g_1, g_2) - k \leq d_Y(\phi(g_1), \phi(g_2)) \leq k d_X(g_1, g_2) + k$
(2) $\frac{1}{k} d_Y(h_1, h_2) - k \leq d_X(\theta(h_1), \theta(h_2)) \leq k d_Y(h_1, h_2) + k$
(3) $d_X(g_1, \theta \circ \phi(g_1)) \leq k$
(4) $d_Y(h_1, \phi \circ \theta(h_1)) \leq k$

By possibly increasing the constant k, we may also assume that $k > 2$ and that $\phi(\epsilon_G) = \epsilon_H$ and $\theta(\epsilon_H) = \epsilon_G$, where ϵ_G and ϵ_H are the identity elements of the groups G and H, respectively.

We extend the functions ϕ and θ to functions $\widetilde{\phi} : G \times A^* \to B^*$ and $\widetilde{\theta} : H \times B^* \to A^*$ as follows. Let $\tilde{A} \subset A$ be a subset containing exactly one element for each inverse pair $a, a^{-1} \in A$. Given a pair $(g, a) \in G \times \tilde{A}$, using property (1) above we let $\widetilde{\phi}(g, a)$ be (a choice of) a nonempty word of length at most $2k$ labeling a path in the Cayley graph Y^1 from the vertex $\phi(g)$ to the vertex $\phi(ga)$ (in the case that $\phi(g) = \phi(ga)$, we can choose $\widetilde{\phi}(g, a)$ to be the nonempty word bb^{-1} for some choice of $b \in B$). We also define $\widetilde{\phi}(g, a^{-1}) := \widetilde{\phi}(ga^{-1}, a)^{-1}$. Then for any $w = a_1 \cdots a_m$ with each $a_i \in A$, define $\widetilde{\phi}(g, w)$ to be the concatenation $\widetilde{\phi}(g, w) := \widetilde{\phi}(g, a_1) \cdots \widetilde{\phi}(ga_1 \cdots a_{m-1}, a_m)$. Note that for $w \in A^*$:

(5) the word lengths satisfy $l(w) \leq l(\widetilde{\phi}(g, w)) \leq 2kl(w)$, and

(6) the word $\widetilde{\phi}(\epsilon_G, w)$ represents the element $\phi(w)$ in H.

The function $\widetilde{\theta}$ is defined analogously.

Using Propositions 3.4 and 3.6, we will prove this theorem using relaxed tame filling inequalities via disk homotopies. For the group G with presentation $\mathcal{P}$, fix a collection $\mathcal{D} = \{(\Delta_w, \Phi_w) \mid w \in A^*, w =_G \epsilon_G\}$ of van Kampen diagrams and associated disk homotopies, such that all of the Φ_w are intrinsically f^i-tame or all Φ_w are extrinsically f^e-tame, where $f^i, f^e : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ are nondecreasing functions.

Case A. Suppose that G is a finite group.

In this case, H is also finite. Let $\mathcal{F}$ be a (finite) collection of van Kampen diagrams over $\mathcal{P}'$, one for each word over B of length at most $|H|$ that represents ϵ_H. Now given any word u over B with $u =_H \epsilon_H$, we will construct a van Kampen diagram for u with intrinsic diameter at most $|H| + \max\{idiam(\Delta) \mid \Delta \in \mathcal{F}\}$, as follows. Start with a planar 1-complex that is a line segment consisting of an edge path labeled by the word u starting at a basepoint $*$; that is, we start with a van Kampen diagram for the word uu^{-1}. Write $u = u_1' u_1'' u_1'''$ where $u_1'' =_H \epsilon_H$ and no proper prefix of $u_1' u_1''$ contains a subword that represents ϵ_H. Note that $l(u_1' u_1'') \leq |H|$. We identify the vertices in the van Kampen diagram at the start and end of the boundary path labeled u_1'', and fill in this loop with the van Kampen diagram from $\mathcal{F}$ for this word. We now have a van Kampen diagram for the word uu_1^{-1} where $u_1 := u_1' u_1'''$. We then begin again, and write $u_1 = u_2' u_2'' u_2'''$ where $u_2'' =_H \epsilon_H$ and no proper prefix of $u_2' u_2''$ contains a subword representing the identity. Again we identify the vertices at the start and end of the word u_2'' in the boundary of the diagram, and fill in this loop with the diagram from $\mathcal{F}$ for this word, to obtain a van Kampen diagram for the word uu_2^{-1} where $u_2 := u_2' u_2'''$. Repeating this process, since at each step the length of u_i strictly decreases, we eventually obtain a word $u_k = u_k''$. Identifying the endpoints of this word and filling in the resulting loop with the van Kampen diagram in $\mathcal{F}$ yields a van Kampen diagram Δ_u' for u. See Figure 10 for an illustration of this procedure. At each step, the maximum distance from the basepoint $*$ to any vertex in the van Kampen diagram included from $\mathcal{F}$ is at most $|H| + \max\{idiam(\Delta) \mid \Delta \in \mathcal{F}\}$, because this subdiagram is attached at the endpoint of a path starting at $*$ and labeled by the word u_i' of length less than $|H|$. At the end of this process, every vertex of the final diagram lies on one of these subdiagrams. Hence we obtain the required intrinsic diameter bound.

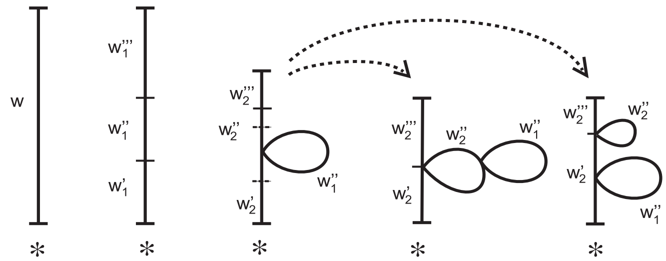

FIGURE 10. Building Δ'_u in the finite group case

Let Φ'_u be any disk homotopy of the diagram Δ'_u. Then the collection $\{(\Delta'_u, \Phi'_u)\}$ of van Kampen diagrams and disk homotopies H over $\mathcal{P}'$ satisfies the property that each homotopy Φ'_u is intrinsically f-tame for the constant function $f(n) \equiv |H| + \max\{idiam(\Delta) \mid \Delta \in \mathcal{F}\} + \frac{1}{2}$, since this constant is an upper bound for the coarse distance from the basepoint to every point of Δ'_u. Hence H also satisfies an intrinsic relaxed tame filling inequality with respect to the function $f^i + f$, which is Lipschitz equivalent to f^i.

Similarly, since the extrinsic diameter of every van Kampen diagram in this collection (or, indeed, any other van Kampen diagram) is at most $|H|$, the pair $(H, \mathcal{P}')$ satisfies an extrinsic tame filling inequality for the constant function $|H| + \frac{1}{2}$, and so also satisfies an extrinsic relaxed tame filling inequality for the function $n \to f^e(n) + |H| + \frac{1}{2}$.

Case B. Suppose that G is an infinite group.

The group H is also infinite, and so the functions f^i and f^e must grow at least linearly, in this case. In particular, we have $f^i(n) \geq n - \zeta - 1$ and $f^e(n) \geq n - \zeta - 1$ for all $n \in \mathbb{N}[\frac{1}{4}]$, where $\zeta = \max\{l(r) \mid r \in R\}$ is the maximum length of a relator in the presentation $\mathcal{P}$.

Now suppose that u' is any word in B^* with $u' =_H \epsilon_H$. We will construct a van Kampen diagram for u', following the method of [4, Theorem 9.1]. At each of the four successive steps, we obtain a van Kampen diagram for a specific word; we will also keep track of homotopies and analyze their tameness, in order to finish with a diagram and disk homotopy for u'.

Step I. For $u := \widetilde{\theta}(\epsilon_H, u') \in A^*$: Note (6) implies that the word $u =_G \theta(u') =_G \theta(\epsilon_H) =_G \epsilon_G$, and so the collection $\mathcal{D}$ contains a van Kampen diagram Δ_u for u and an associated disk homotopy $\Phi_u : C_{l(u)} \times [0,1] \to \Delta_u$. Note that Φ_u is intrinsically $f_1^i := f^i$-tame or extrinsically $f_1^e := f^e$-tame.

Step II. For $z'' := \widetilde{\phi}(\epsilon_G, u) = \widetilde{\phi}(\epsilon_G, \widetilde{\theta}(\epsilon_H, u')) \in B^*$: We build a finite, planar, contractible, combinatorial 2-complex Ω from Δ_u as follows. As usual, let $\pi_{\Delta_u} : \Delta_u \to X$ be the canonical map taking the basepoint $*$ of Δ_u to ϵ_G. Given any edge e in Δ_u, choose a direction, and hence a label a_e, for e, and let v_1 be the initial vertex of e. Replace e with a directed edge path $\hat{e}$ labeled by the (nonempty) word $\widetilde{\phi}(\pi_{\Delta_u}(v_1), a_e)$. Repeating this for every edge of the complex Δ_u results in the 2-complex Ω.

Note that Ω is a van Kampen diagram for the word $z'' := \widetilde{\phi}(\epsilon_G, u) = \widetilde{\phi}(\epsilon_G, \widetilde{\theta}(\epsilon_H, u')) \in B^*$ with respect to the presentation $\mathcal{P}'' = \langle B \mid S \cup S'' \rangle$ of H, where S'' is the set of all nonempty

words over B of length at most $2k\zeta$ that represent ϵ_H. Let Y'' be the Cayley complex for $\mathcal{P}''$ and as usual, let $\pi_\Omega : \Omega \to Y''$ be the canonical map.

Using the fact that the only difference between Δ_u and Ω is a replacement of edges by edge paths, we define $\alpha : \Delta_u \to \Omega$ to be the continuous map taking each vertex and each interior point of a 2-cell of Δ_u to the same point of Ω, and taking each edge e to the corresponding edge path $\hat{e}$.

Writing $u = a_1 \cdots a_m$ with each $a_i \in A$, then $z'' = c_{1,1} \cdots c_{1,j_1} \cdots c_{m,1} \cdots c_{m,j_m}$ where each $c_{i,j} \in B$ and $c_{i,1} \cdots c_{i,j_i}$ is the nonempty word labeling the edge path $\widehat{e_i}$ of $\partial\Omega$ that is the image under α of the i-th edge of the boundary path of Δ_u. Recall that $C_{l(u)}$ is the circle S^1 with a 1-complex structure of $l(u)$ vertices and edges. Let the 1-complex $C_{l(z'')}$ be a refinement of the complex $C_{l(u)}$, so that the i-th edge of $C_{l(u)}$ is replaced by $j_i \geq 1$ edges for each i, and let $\hat{\alpha} : C_{l(z'')} \to C_{l(u)}$ be the identity on the underlying circle. Finally, define the map $\omega : C_{l(z'')} \times [0,1] \to \Omega$ by $\omega := \alpha \circ \Phi_u \circ (\hat{\alpha} \times id_{[0,1]})$. This map ω satisfies conditions (d1)-(d2) of the definition of disk homotopy.

Next we analyze the intrinsic tameness of ω. Again since in this step we have only replaced edges by nonempty edge paths of length at most $2k$, for each vertex v in Δ_u we have $\widetilde{d}_{\Delta_u}(*, v) \leq \widetilde{d}_\Omega(*, \alpha(v)) \leq 2k\widetilde{d}_{\Delta_u}(*, v)$. For a point q in the interior of an edge of Δ_u, let v be a vertex in the same closed cell; then $|\widetilde{d}_{\Delta_u}(*, q) - \widetilde{d}_{\Delta_u}(*, v)| < 1$ and $|\widetilde{d}_\Omega(*, \alpha(q)) - \widetilde{d}_\Omega(*, \alpha(v))| < 2k$. For a point q in the interior of a 2-cell of Δ_u, let v be a vertex in the closure of this cell with $\widetilde{d}_{\Delta_u}(*, v) \leq \widetilde{d}_{\Delta_u}(*, q) + 1$. Then $\alpha(v)$ is a vertex in the closure of the open 2-cell of Ω containing $\alpha(q)$, and the boundary path of this cell has length at most $2k\zeta$. That is, $|\widetilde{d}_{\Delta_u}(*, q) - \widetilde{d}_{\Delta_u}(*, v)| < 1$ and $|\widetilde{d}_\Omega(*, \alpha(q)) - \widetilde{d}_\Omega(*, \alpha(v))| < 2k\zeta$. Thus for all $q \in \Delta_u$, we have $\widetilde{d}_{\Delta_u}(*, q) \leq \widetilde{d}_\Omega(*, \alpha(q)) \leq 2k\widetilde{d}_{\Delta_u}(*, q) + 4k + 2k\zeta$.

Now suppose that p is any point in $C_{l(z'')}$ and $0 \leq s < t \leq 1$. Combining the inequalities above with the f_1^i-tame property of Φ_u and the fact that f_1^i is nondecreasing yields

$$\begin{aligned}
\widetilde{d}_\Omega(*, \omega(p, s)) &= \widetilde{d}_\Omega(*, \alpha(\Phi_u(\hat{\alpha}(p), s)) \\
&\leq 2k\widetilde{d}_{\Delta_u}(*, \Phi_u(\hat{\alpha}(p), s)) + 4k + 2k\zeta \\
&\leq 2kf_1^i(\widetilde{d}_{\Delta_u}(*, \Phi_u(\hat{\alpha}(p), t))) + 4k + 2k\zeta \\
&\leq 2kf_1^i(\widetilde{d}_\Omega(*, \alpha(\Phi_u(\hat{\alpha}(p), t)))) + 4k + 2k\zeta \\
&= 2kf_1^i(\widetilde{d}_\Omega(*, \omega(p, t)) + 4k + 2k\zeta \ .
\end{aligned}$$

Hence ω is intrinsically f_2^i-tame for the nondecreasing function $f_2^i(n) := 2kf_1^i(n) + 4k + 2k\zeta$.

In the last part of Step II, we analyze the extrinsic tameness of ω. For any vertex v in Δ_u, let w_v be a word labeling a path in Δ_u from $*$ to v. Using note (6) above, we have $\phi(\pi_{\Delta_u}(v)) =_H \phi(w_v) =_H \widetilde{\phi}(\epsilon_G, w_v) = \pi_\Omega(\alpha(v))$, by our construction of Ω. The quasi-isometry property (1) then gives

$$\frac{1}{k}d_X(\epsilon_G, \pi_{\Delta_u}(v)) - k \leq d_Y(\epsilon_H, \phi(\pi_{\Delta_u}(v))) = d_Y(\epsilon_H, \pi_\Omega(\alpha(v))) \leq kd_X(\epsilon_G, \pi_{\Delta_u}(v)) + k \ .$$

Since the generating sets of the presentations $\mathcal{P}'$ and $\mathcal{P}''$ of H are the same, the Cayley graphs and their path metrics $d_Y = d_{Y''}$ are also the same. As in the intrinsic case above, for

a point q in the interior of an edge or 2-cell of Δ_u, there is a vertex v in the same closed cell with $|\widetilde{d}_X(\epsilon_G,\pi_{\Delta_u}(q)) - \widetilde{d}_X(\epsilon_G,\pi_{\Delta_u}(v))| < 1$ and $|\widetilde{d}_{Y''}(\epsilon_G,\pi_\Omega(\alpha(q))) - \widetilde{d}_{Y''}(\epsilon_G,\pi_\Omega(\alpha(v)))| < 2k(\zeta+1)$. Then for all $q \in \Delta_u$, we have

$$\begin{aligned}\widetilde{d}_X(\epsilon_G,\pi_{\Delta_u}(q)) &\leq k\widetilde{d}_{Y''}(\epsilon_H,\pi_\Omega(\alpha(q))) + 2k^2\zeta + 3k^2 + 1 \ , \text{ and} \\ \widetilde{d}_{Y''}(\epsilon_H,\pi_\Omega(\alpha(q))) &\leq k\widetilde{d}_X(\epsilon_G,\pi_{\Delta_u}(q)) + 4k + 2k\zeta \ .\end{aligned}$$

Now suppose that p is any point in $C_{l(z'')}$ and $0 \leq s < t \leq 1$. Then

$$\begin{aligned}\widetilde{d}_{Y''}(\epsilon_H,\pi_\Omega(\omega(p,s))) &= \widetilde{d}_{Y''}(\epsilon_H,\pi_\Omega(\alpha(\Phi_u(\hat\alpha(p),s)))) \\ &\leq k\widetilde{d}_X(\epsilon_G,\pi_{\Delta_u}(\Phi_u(\hat\alpha(p),s))) + 4k + 2k\zeta \\ &\leq kf_1^e(\widetilde{d}_X(\epsilon_G,\pi_{\Delta_u}(\Phi_u(\hat\alpha(p),t)))) + 4k + 2k\zeta \\ &\leq kf_1^e(k\widetilde{d}_{Y''}(\epsilon_H,\pi_\Omega(\alpha(\Phi_u(\hat\alpha(p),t)))) + 2k^2\zeta + 3k^2 + 1) + 4k + 2k\zeta \\ &= kf_1^e(k\widetilde{d}_{Y''}(\epsilon_H,\pi_\Omega(\omega(p,t)) + 2k^2\zeta + 3k^2 + 1) + 4k + 2k\zeta \ .\end{aligned}$$

Hence ω is extrinsically f_2^e-tame for the nondecreasing function $f_2^e(n) := kf_1^e(kn + 2k^2\zeta + 3k^2 + 1) + 4k + 2k\zeta$.

Step III. For u' over $\mathcal{P}'''$: In this step we construct another finite, planar, contractible, and combinatorial 2-complex $\Lambda_{u'}$ starting from Ω, by adding a "collar" around the outside boundary. Write the word $u' = b_1 \cdots b_n$ with each $b_i \in B$. For each $1 \leq i \leq n-1$, let w_i be a word labeling a geodesic edge path in Y from $\phi(\theta(b_1 \cdots b_i))$ to $b_1 \cdots b_i$; the quasi-isometry inequality in (3) above implies that the length of w_i is at most k. We add to $\Lambda_{u'}$ a vertex x_i and the vertices and edges of a directed edge path p_i labeled by w_i from the vertex v_i to x_i, where v_i is the vertex in $\partial\Delta_u$ at the end of the path $\widetilde{\phi}(\epsilon_G, \widetilde{\theta}(e_H, b_1 \cdots b_i))$ starting at the basepoint. Note that if w_i is the empty word, we identify x_i with the vertex v_i; the path p_i is a constant path at this vertex. Then $* = v_0 = x_0 = x_n$ (and p_0 and p_n are the constant path at this vertex); let this vertex be the basepoint of $\Lambda_{u'}$.

Next we add to $\Lambda_{u'}$ a directed edge $\check{e}_i$ labeled by b_i from the vertex x_{i-1} to the vertex x_i. The path q_i from v_{i-1} to v_i along the boundary of the subcomplex Ω is labeled by the nonempty word $z_i := \widetilde{\phi}(\theta(b_1 \cdots b_{i-1}), \widetilde{\theta}(b_1 \cdots b_{i-1}, b_i))$. If both of the paths p_{i-1}, p_i are constant and the label of path q_i is the single letter b_i, then we identify the edge $\check{e}_i$ with the path q_i. Otherwise, we attach a 2-cell $\check{\sigma}_i$ along the edge circuit following the edge path starting at $\hat{v}_{i-1}$ that traverses the path q_i, the path p_i, the reverse of the edge $\check{e}_i$, and finally the reverse of the path p_{i-1}. See Figure 11 for a picture of the resulting diagram.

Now the complex $\Lambda_{u'}$ is a van Kampen diagram for the original word u', with respect to the presentation $\mathcal{P}''' = \langle B \mid S \cup S''' \rangle$ of H, where S''' is the set of all nonempty words in B^* of length at most $\zeta''' := 2k\zeta + (2k)^2 + 2k + 1$ that represent ϵ_H. (Note that the presentation $\langle B \mid S''' \rangle$ also presents H, and $\Lambda_{u'}$ is also a diagram over this more restricted presentation.) Let Y''' be the corresponding Cayley complex.

We define a disk homotopy $\lambda_{u'} : C_{l(u')} \times [0,1] \to \Lambda_{u'}$ by extending the paths of the homotopy ω on the subcomplex Ω as follows. First we let the cell complex $C_{l(u')}$ be the complex $C_{l(z''))}$ with each subpath in $C_{l(z''))}$ mapping to a path q_i in $\partial\Omega$ replaced by a single edge. From our definitions of $\widetilde{\phi}$ and $\widetilde{\theta}$, each q_i path is labeled by a nonempty word, and so

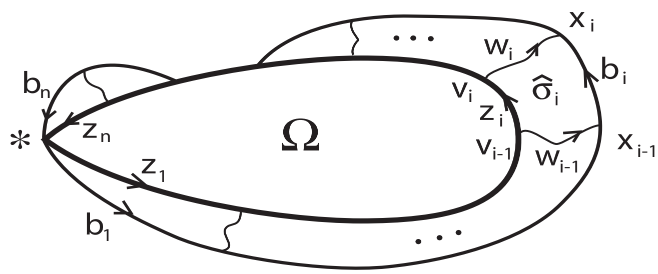

FIGURE 11. The van Kampen diagram $\Lambda_{u'}$

$C_{l(z''))}$ is a refinement of the complex structure $C_{l(u'))}$ on S^1, and we let $\hat{\beta} : C_{l(u'))} \to C_{l(z''))}$ be the identity on the underlying circle. Next define a homotopy $\tilde{\lambda} : C_{l(z'')} \times [0,1] \to \Lambda_{u'}$ as follows. For each $1 \leq i \leq n$, let $\tilde{v}_i$ be the point in S^1 mapped by ω to v_i. Define $\tilde{\lambda}(\tilde{v}_i, t) := \omega(\tilde{v}_i, 2t)$ for $t \in [0, \frac{1}{2}]$, and let $\tilde{\lambda}(\tilde{v}_i, t)$ for $t \in [\frac{1}{2}, 1]$ be a constant speed path along p_i from v_i to x_i. On the interior of the edge $\tilde{e}_i$ from $\tilde{v}_{i-1}$ to $\tilde{v}_i$, define the homotopy $\tilde{\lambda}|_{\tilde{e}_i \times [0,\frac{1}{2}]}$ to follow $\omega|_{\tilde{e}_i \times [0,1]}$ at double speed, and let $\tilde{\lambda}|_{\tilde{e}_i \times [\frac{1}{2},1]}$ go through the 2-cell $\check{\sigma}_i$ (or, if there is no such cell, let this portion of $\tilde{\lambda}$ be constant) from q_i to $\check{e}_i$. Finally, we define the homotopy $\lambda_{u'} : C_{l(u'))} \times [0,1] \to \Lambda_{u'}$ by $\lambda_{u'} := \tilde{\lambda} \circ (\hat{\beta} \times id_{[0,1]})$. This map $\lambda_{u'}$ is a disk homotopy for the diagram $\Lambda_{u'}$.

Next we analyze the intrinsic tameness of $\lambda_{u'}$. Since Ω is a subdiagram of $\Lambda_{u'}$, for any vertex v in Ω, we have $d_{\Lambda_{u'}}(*, v) \leq d_\Omega(*, v)$. Given any edge path β in $\Lambda_{u'}$ from ϵ to q that is not completely contained in the subdiagram Ω, the subpaths of β lying in the "collar" can be replaced by paths along $\partial\Omega$ of length at most a factor of $4k^2$ longer. Then $d_\Omega(*, v) \leq 4k^2 d_{\Lambda_{u'}}(*, v)$. Hence for any point $q \in \Omega$, we have $\widetilde{d}_{\Lambda_{u'}}(*, q) \leq \widetilde{d}_\Omega(*, q) \leq 4k^2 \widetilde{d}_{\Lambda_{u'}}(*, q) + 4k^2 + 1 + \zeta'''$.

Now suppose that p is any point of $C_{l(u')}$ and $0 \leq s < t \leq 1$. If $t \leq \frac{1}{2}$, then the path $\lambda_{u'}(p, \cdot)$ on $[0, t]$ is a reparametrization of $\omega(p, \cdot)$, and so Step II, the fact that f_2^i is nondecreasing, and the inequalities above give

$$\begin{aligned}
\widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p, s)) &\leq \widetilde{d}_\Omega(*, \lambda_{u'}(p, s)) \\
&\leq f_2^i(\widetilde{d}_\Omega(*, \lambda_{u'}(p, t))) \\
&\leq f_2^i(4k^2 \widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p, t)) + 4k^2 + 1 + \zeta''')
\end{aligned}$$

If $t > \frac{1}{2}$ and $s \leq \frac{1}{2}$, then we have $\widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p, s)) \leq f_2^i(4k^2 \widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p, \frac{1}{2})) + 4k^2 + 1 + \zeta''')$ and $|\widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p, t)) - \widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p, \frac{1}{2}))| < \zeta''' + 1$, so

$$\widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p, s)) \leq f_2^i(4k^2(\widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p, t)) + \zeta''' + 1) + 4k^2 + 1 + \zeta''').$$

If $s > \frac{1}{2}$, then

$$\begin{aligned}\widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p,s)) &\leq \widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p,t)) + \zeta''' + 1 \\ &\leq f_2^i(\widetilde{d}_{\Lambda_{u'}}(*, \lambda_{u'}(p,t))) + \zeta''' + 1,\end{aligned}$$

where the latter inequality follows from the fact that $n \leq f^i(n) + \zeta + 1 \leq f_2^i(n)$ for this infinite group case. Then $\lambda_{u'}$ is intrinsically f_3^i-tame for the function $f_3^i(n) := f_2^i(4k^2n + 8k^2 + 1 + (4k^2+1)\zeta''')$.

We note that we have now completed the proof of Theorem 7.1 in the intrinsic case: The collection $\{(\Lambda_{u'}, \lambda_{u'}) \mid u' \in B^*, u' =_H \epsilon_H\}$ of van Kampen diagrams and disk homotopies over the presentation $\mathcal{P}''' = \langle B \mid S''' \rangle$ implies an intrinsic relaxed tame filling inequality for the function f_3^i, which is Lipschitz equivalent to f^i.

The analysis of the extrinsic tameness in this step is simplified by the fact that for all $q \in \Omega$, we have $\widetilde{d}_{Y''}(\epsilon_H, \pi_\Omega(q)) = \widetilde{d}_{Y'''}(\epsilon_H, \pi_{\Lambda_{u'}}(q))$, since the 1-skeleta of Y'' and Y''' are determined by the generating sets of the presentations $\mathcal{P}''$ and $\mathcal{P}'''$, which are the same. A similar argument to those above shows that $\lambda_{u'}$ is extrinsically f_3^e-tame for the function $f_3^e(n) := f_2^e(n + \zeta''' + 1) + \zeta + 1$.

Step IV. For u' over $\mathcal{P}'$: Finally, we turn to building a van Kampen diagram $\Delta'_{u'}$ for u' over the original presentation $\mathcal{P}'$. For each nonempty word w over B of length at most ζ''' satisfying $w =_H \epsilon_H$, let Δ'_w be a fixed choice of van Kampen diagram for w with respect to the presentation $\mathcal{P}'$ of H, and let $\mathcal{F}$ be the (finite) collection of these diagrams. A diagram $\Delta'_{u'}$ over the presentation $\mathcal{P}'$ is built by replacing 2-cells of $\Lambda_{u'}$, proceeding through the 2-cells of $\Lambda_{u'}$ one at a time. Let τ be a 2-cell of $\Lambda_{u'}$, and let $*_\tau$ be a choice of basepoint vertex in $\partial\tau$. Let x be the word labeling the path $\partial\tau$ starting at $*_\tau$ and reading counterclockwise. Since $l(x) \leq L$, there is an associated van Kampen diagram $\Delta'_\tau = \Delta'_x$ in the collection $\mathcal{F}$. Note that although $\Lambda_{u'}$ is a combinatorial 2-complex, and so the cell τ is a polygon, the boundary label x may not be freely or cyclically reduced. The van Kampen diagram Δ'_x may not be a polygon, but instead a collection of polygons connected by edge paths, and possibly with edge path "tendrils". We replace the 2-cell τ with a copy Δ'_τ of the van Kampen diagram Δ'_x, identifying the boundary edge labels as needed, obtaining another planar diagram. Repeating this for each 2-cell of of the resulting complex at each step, results in the van Kampen diagram $\Delta'_{u'}$ for u' with respect to $\mathcal{P}'$.

From the process of constructing $\Delta'_{u'}$ from Λ, for each 2-cell τ there is a continuous onto map $\tau \to \Delta'_\tau$ preserving the boundary edge path labeling, and so there is an induced continuous surjection $\gamma : \Lambda_{u'} \to \Delta'_{u'}$. Note that the boundary edge paths of $\Lambda_{u'}$ and $\Delta'_{u'}$ are the same. Then the composition $\Phi'_{u'} := \gamma \circ \lambda_{u'} : C_{l(u')} \times [0,1] \to \Delta'_{u'}$ is a disk homotopy.

To analyze the extrinsic tameness, we first note that for all points $\hat{q} \in \Lambda^1_{u'}$, the image $\pi_{\Lambda_{u'}}(\hat{q})$ in Y''' and the image $\pi_{\Delta'_{u'}}(\gamma(\hat{q}))$ in Y are the same point in the 1-skeleta $Y^1 = (Y''')^1$, and so $\widetilde{d}_{Y'''}(\epsilon_H, \pi_{\Lambda_{u'}}(\hat{q})) = \widetilde{d}_Y(\pi_{\Delta'_{u'}}(\gamma(\hat{q})))$. Let $M := 2\max\{\widetilde{d}_\Delta(*, r) \mid \Delta \in \mathcal{F}, r \in \Delta\}$.

Suppose that p is any point in $C_{l(u')}$ and $0 \leq s < t \leq 1$. If $\lambda_{u'}(p,s) \in \Lambda^1_{u'}$, then define $s' := s$; otherwise, let $0 \leq s' < s$ satisfy $\lambda_{u'}(p,s') \in \Lambda^1_{u'}$ and $\lambda_{u'}(p,(s',s])$ is a subset of a single open 2-cell of $\Lambda_{u'}$. Similarly, if $\lambda_{u'}(p,t) \in \Lambda^1_{u'}$, then define $t' := t$, and otherwise, let

$t < t' \leq 1$ satisfy $\lambda_{u'}(p,t') \in \Lambda^1_{u'}$ and $\lambda_{u'}(p,[t,t'))$ is a subset of a single open 2-cell of $\Lambda_{u'}$. Then

$$\begin{aligned}
\widetilde{d}_Y(\epsilon_H, \pi_{\Delta'_{u'}}(\Phi_{u'}(p,s))) &= \widetilde{d}_Y(\epsilon_H, \pi_{\Delta'_{u'}}(\gamma(\lambda_{u'}(p,s)))) \\
&\leq \widetilde{d}_Y(\epsilon_H, \pi_{\Delta'_{u'}}(\gamma(\lambda_{u'}(p,s')))) + M \\
&= \widetilde{d}_{Y'''}(\epsilon_H, \pi_{\Lambda_{u'}}(\lambda_{u'}(p,s'))) + M \\
&\leq f_3^e(\widetilde{d}_{Y'''}(\epsilon_H, \pi_{\Lambda_{u'}}(\lambda_{u'}(p,t')))) + M \\
&= f_3^e(\widetilde{d}_Y(\epsilon_H, \pi_{\Delta'_{u'}}(\gamma(\lambda_{u'}(p,t'))))) + M \\
&\leq f_3^e(\widetilde{d}_Y(\epsilon_H, \pi_{\Delta'_{u'}}(\gamma(\lambda_{u'}(p,t)))) + M) + M.
\end{aligned}$$

Therefore $\Phi'_{u'}$ is extrinsically f_4^e-tame, for the function $f_4(n) := f_4(n+M) + M$. Since the functions f_j^e and f_{j+1}^e are Lipschitz equivalent for all j, then f_4^e is Lipschitz equivalent to f^e.

Now the collection $\{(\Delta'_{u'}, \Phi'_{u'}) \mid u' \in B^*, u' =_H \epsilon_H\}$ of van Kampen diagrams and disk homotopies yields an extrinsic relaxed tame filling inequality for the pair $(H, \mathcal{P}')$ with respect to a function that is Lipschitz equivalent to f^e. $\square$

The obstruction to applying Step IV of the above proof in the intrinsic case stems from the fact that the map $\gamma : \Lambda_{u'} \to \Delta'_{u'}$ behaves well with respect to extrinsic coarse distance, but may not behave well with respect to intrinsic coarse distance. The latter results because the replacement of a 2-cell τ of $\Lambda_{u'}$ with a van Kampen diagram Δ'_τ can result in the identification of vertices of $\Lambda_{u'}$.

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